



11076CH05

# COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics.* – GAUSS ❖

## 4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation  $x^2 + 1 = 0$  has no real solution as  $x^2 + 1 = 0$  gives  $x^2 = -1$  and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation  $x^2 = -1$ . In fact, the main objective is to solve the equation  $ax^2 + bx + c = 0$ , where  $D = b^2 - 4ac < 0$ , which is not possible in the system of real numbers.



W. R. Hamilton  
(1805-1865)

## 4.2 Complex Numbers

Let us denote  $\sqrt{-1}$  by the symbol  $i$ . Then, we have  $i^2 = -1$ . This means that  $i$  is a solution of the equation  $x^2 + 1 = 0$ .

A number of the form  $a + ib$ , where  $a$  and  $b$  are real numbers, is defined to be a complex number. For example,  $2 + i3$ ,  $(-1) + i\sqrt{3}$ ,  $4 + i\left(\frac{-1}{11}\right)$  are complex numbers.

For the complex number  $z = a + ib$ ,  $a$  is called the *real part*, denoted by  $\text{Re } z$  and  $b$  is called the *imaginary part* denoted by  $\text{Im } z$  of the complex number  $z$ . For example, if  $z = 2 + i5$ , then  $\text{Re } z = 2$  and  $\text{Im } z = 5$ .

Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  are equal if  $a = c$  and  $b = d$ .

**Example 1** If  $4x + i(3x - y) = 3 + i(-6)$ , where  $x$  and  $y$  are real numbers, then find the values of  $x$  and  $y$ .

**Solution** We have

$$4x + i(3x - y) = 3 + i(-6) \quad \dots (1)$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3, 3x - y = -6,$$

which, on solving simultaneously, give  $x = \frac{3}{4}$  and  $y = \frac{33}{4}$ .

### 4.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

**4.3.1 Addition of two complex numbers** Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers. Then, the sum  $z_1 + z_2$  is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$ , which is again a complex number.

For example,  $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i8$

The addition of complex numbers satisfy the following properties:

- (i) *The closure law* The sum of two complex numbers is a complex number, i.e.,  $z_1 + z_2$  is a complex number for all complex numbers  $z_1$  and  $z_2$ .
- (ii) *The commutative law* For any two complex numbers  $z_1$  and  $z_2$ ,  $z_1 + z_2 = z_2 + z_1$
- (iii) *The associative law* For any three complex numbers  $z_1, z_2, z_3$ ,  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ .
- (iv) *The existence of additive identity* There exists the complex number  $0 + i0$  (denoted as 0), called the *additive identity* or the *zero complex number*, such that, for every complex number  $z$ ,  $z + 0 = z$ .
- (v) *The existence of additive inverse* To every complex number  $z = a + ib$ , we have the complex number  $-a + i(-b)$  (denoted as  $-z$ ), called the *additive inverse* or *negative of z*. We observe that  $z + (-z) = 0$  (the additive identity).

**4.3.2 Difference of two complex numbers** Given any two complex numbers  $z_1$  and  $z_2$ , the difference  $z_1 - z_2$  is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example,  $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and  $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

**4.3.3 Multiplication of two complex numbers** Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers. Then, the product  $z_1 z_2$  is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example,  $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

- (i) **The closure law** The product of two complex numbers is a complex number, the product  $z_1 z_2$  is a complex number for all complex numbers  $z_1$  and  $z_2$ .
- (ii) **The commutative law** For any two complex numbers  $z_1$  and  $z_2$ ,  

$$z_1 z_2 = z_2 z_1$$
- (iii) **The associative law** For any three complex numbers  $z_1, z_2, z_3$ ,  

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$
- (iv) **The existence of multiplicative identity** There exists the complex number  $1 + i0$  (denoted as 1), called the *multiplicative identity* such that  $z \cdot 1 = z$ , for every complex number  $z$ .
- (v) **The existence of multiplicative inverse** For every non-zero complex number  $z = a + ib$  or  $a + bi$  ( $a \neq 0, b \neq 0$ ), we have the complex number

$\frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2}$  (denoted by  $\frac{1}{z}$  or  $z^{-1}$ ), called the *multiplicative inverse* of  $z$  such that

$$z \cdot \frac{1}{z} = 1 \text{ (the multiplicative identity).}$$

- (vi) **The distributive law** For any three complex numbers  $z_1, z_2, z_3$ ,
  - (a)  $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
  - (b)  $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

**4.3.4 Division of two complex numbers** Given any two complex numbers  $z_1$  and  $z_2$ ,

where  $z_2 \neq 0$ , the quotient  $\frac{z_1}{z_2}$  is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let  $z_1 = 6 + 3i$  and  $z_2 = 2 - i$

Then 
$$\frac{z_1}{z_2} = \left( (6 + 3i) \times \frac{1}{2 - i} \right) = (6 + 3i) \left( \frac{2}{2^2 + (-1)^2} + i \frac{-(-1)}{2^2 + (-1)^2} \right)$$

$$= (6 + 3i) \left( \frac{2+i}{5} \right) = \frac{1}{5} [12 - 3 + i(6+6)] = \frac{1}{5} (9 + 12i)$$

**4.3.5 Power of  $i$**  we know that

$$i^3 = i^2 i = (-1) i = -i, \quad i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^5 = (i^2)^2 i = (-1)^2 i = i, \quad i^6 = (i^2)^3 = (-1)^3 = -1, \text{ etc.}$$

Also, we have  $i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer  $k$ ,  $i^{4k} = 1$ ,  $i^{4k+1} = i$ ,  $i^{4k+2} = -1$ ,  $i^{4k+3} = -i$

#### 4.3.6 The square roots of a negative real number

Note that  $i^2 = -1$  and  $(-i)^2 = i^2 = -1$

Therefore, the square roots of  $-1$  are  $i, -i$ . However, by the symbol  $\sqrt{-1}$ , we would mean  $i$  only.

Now, we can see that  $i$  and  $-i$  both are the solutions of the equation  $x^2 + 1 = 0$  or  $x^2 = -1$ .

Similarly  $(\sqrt{3}i)^2 = (\sqrt{3})^2 i^2 = 3(-1) = -3$

$$(-\sqrt{3}i)^2 = (-\sqrt{3})^2 i^2 = -3$$

Therefore, the square roots of  $-3$  are  $\sqrt{3}i$  and  $-\sqrt{3}i$ .

Again, the symbol  $\sqrt{-3}$  is meant to represent  $\sqrt{3}i$  only, i.e.,  $\sqrt{-3} = \sqrt{3}i$ .

Generally, if  $a$  is a positive real number,  $\sqrt{-a} = \sqrt{a} \sqrt{-1} = \sqrt{a}i$ ,

We already know that  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$  for all positive real number  $a$  and  $b$ . This result also holds true when either  $a > 0, b < 0$  or  $a < 0, b > 0$ . What if  $a < 0, b < 0$ ? Let us examine.

Note that

$$i^2 = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)} \text{ (by assuming } \sqrt{a} \times \sqrt{b} = \sqrt{ab} \text{ for all real numbers)}$$

$$= \sqrt{1} = 1, \text{ which is a contradiction to the fact that } i^2 = -1.$$

Therefore,  $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$  if both  $a$  and  $b$  are negative real numbers.

Further, if any of  $a$  and  $b$  is zero, then, clearly,  $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$ .

**4.3.7 Identities** We prove the following identity

$$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2, \text{ for all complex numbers } z_1 \text{ and } z_2.$$

**Proof** We have,  $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$ ,

$$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2 \quad (\text{Distributive law})$$

$$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 \quad (\text{Distributive law})$$

$$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2 \quad (\text{Commutative law of multiplication})$$

$$= z_1^2 + 2z_1z_2 + z_2^2$$

Similarly, we can prove the following identities:

$$(i) \quad (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$$

$$(ii) \quad (z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$$

$$(iii) \quad (z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$$

$$(iv) \quad z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

**Example 2** Express the following in the form of  $a + bi$ :

$$(i) \quad (-5i) \left( \frac{1}{8}i \right)$$

$$(ii) \quad (-i)(2i) \left( -\frac{1}{8}i \right)^3$$

**Solution** (i)  $(-5i) \left( \frac{1}{8}i \right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

$$(ii) \quad (-i)(2i) \left( -\frac{1}{8}i \right)^3 = 2 \times \frac{1}{8 \times 8 \times 8} \times i^5 = \frac{1}{256} (i^2)^2 i = \frac{1}{256} i.$$

**Example 3** Express  $(5 - 3i)^3$  in the form  $a + ib$ .

**Solution** We have,  $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5 (3i)^2 - (3i)^3$   
 $= 125 - 225i - 135 + 27i = -10 - 198i$ .

**Example 4** Express  $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$  in the form of  $a + ib$

**Solution** We have,  $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i) = (-\sqrt{3} + \sqrt{2}i)(2\sqrt{3} - i)$   
 $= -6 + \sqrt{3}i + 2\sqrt{6}i - \sqrt{2}i^2 = (-6 + \sqrt{2}) + \sqrt{3}(1 + 2\sqrt{2})i$

#### 4.4 The Modulus and the Conjugate of a Complex Number

Let  $z = a + ib$  be a complex number. Then, the modulus of  $z$ , denoted by  $|z|$ , is defined to be the non-negative real number  $\sqrt{a^2 + b^2}$ , i.e.,  $|z| = \sqrt{a^2 + b^2}$  and the conjugate of  $z$ , denoted as  $\bar{z}$ , is the complex number  $a - ib$ , i.e.,  $\bar{z} = a - ib$ .

For example,  $|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$ ,  $|2 - 5i| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ ,

and  $\overline{3 + i} = 3 - i$ ,  $\overline{2 - 5i} = 2 + 5i$ ,  $\overline{-3i - 5} = 3i - 5$

Observe that the multiplicative inverse of the non-zero complex number  $z$  is given by

$$z^{-1} = \frac{1}{a + ib} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or  $z \bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two complex numbers  $z_1$  and  $z_2$ , we have

$$(i) \quad |z_1 z_2| = |z_1| |z_2| \quad (ii) \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \text{ provided } |z_2| \neq 0$$

$$(iii) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \quad (iv) \quad \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \quad (v) \quad \overline{\left( \frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2} \text{ provided } z_2 \neq 0.$$

**Example 5** Find the multiplicative inverse of  $2 - 3i$ .

**Solution** Let  $z = 2 - 3i$

Then  $\bar{z} = 2 + 3i$  and  $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of  $2 - 3i$  is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$\begin{aligned} z^{-1} &= \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)} \\ &= \frac{2+3i}{2^2 - (3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i \end{aligned}$$

**Example 6** Express the following in the form  $a + ib$

(i)  $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$

(ii)  $i^{-35}$

**Solution** (i) We have,  $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-(\sqrt{2}i)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i$$

(ii)  $i^{-35} = \frac{1}{i^{35}} = \frac{1}{(i^2)^{17}i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$

### EXERCISE 4.1

Express each of the complex number given in the Exercises 1 to 10 in the form  $a + ib$ .

1.  $(5i)\left(-\frac{3}{5}i\right)$

2.  $i^9 + i^{19}$

3.  $i^{-39}$

4.  $3(7 + i7) + i(7 + i7)$       5.  $(1 - i) - (-1 + i6)$
6.  $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$       7.  $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$
8.  $(1 - i)^4$       9.  $\left(\frac{1}{3} + 3i\right)^3$       10.  $\left(-2 - \frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11.  $4 - 3i$       12.  $\sqrt{5} + 3i$       13.  $-i$
14. Express the following expression in the form of  $a + ib$  :

$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

#### 4.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers  $(x, y)$ , we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the  $x$ -axis and the  $y$ -axis. The complex number  $x + iy$  which corresponds to the ordered pair  $(x, y)$  can be represented geometrically as the unique point  $P(x, y)$  in the XY-plane and vice-versa.

Some complex numbers such as  $2 + 4i$ ,  $-2 + 3i$ ,  $0 + 1i$ ,  $2 + 0i$ ,  $-5 - 2i$  and  $1 - 2i$  which correspond to the ordered pairs  $(2, 4)$ ,  $(-2, 3)$ ,  $(0, 1)$ ,  $(2, 0)$ ,  $(-5, -2)$ , and  $(1, -2)$ , respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1.

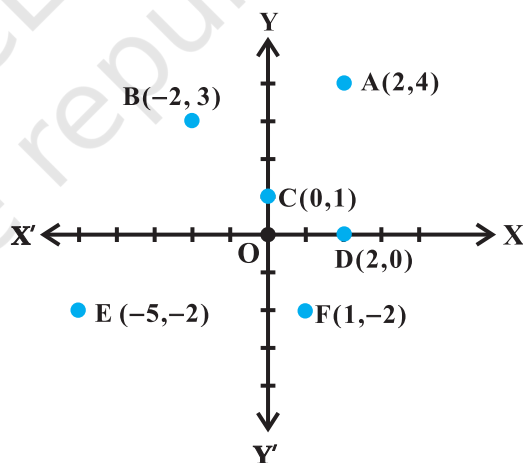


Fig 4.1

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.



Obviously, in the Argand plane, the modulus of the complex number  $x + iy = \sqrt{x^2 + y^2}$  is the distance between the point  $P(x, y)$  and the origin  $O(0, 0)$  (Fig 4.2). The points on the  $x$ -axis corresponds to the complex numbers of the form  $a + i0$  and the points on the  $y$ -axis corresponds to the complex numbers of the form

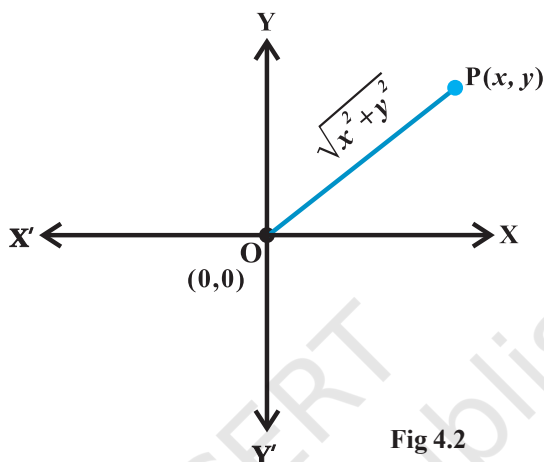


Fig 4.2

$0 + ib$ . The  $x$ -axis and  $y$ -axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number  $z = x + iy$  and its conjugate  $z = x - iy$  in the Argand plane are, respectively, the points  $P(x, y)$  and  $Q(x, -y)$ .

Geometrically, the point  $(x, -y)$  is the mirror image of the point  $(x, y)$  on the real axis (Fig 4.3).

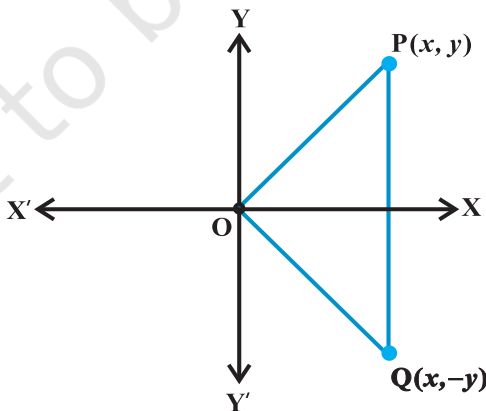


Fig 4.3

### Miscellaneous Examples

**Example 7** Find the conjugate of  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ .

**Solution** We have,  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$\begin{aligned} &= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\ &= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i \end{aligned}$$

Therefore, conjugate of  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$  is  $\frac{63}{25} + \frac{16}{25}i$ .

**Example 8** If  $x + iy = \frac{a+ib}{a-ib}$ , prove that  $x^2 + y^2 = 1$ .

**Solution** We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2 - b^2 + 2abi}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$$

So that,  $x - iy = \frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2 - b^2)^2}{(a^2 + b^2)^2} + \frac{4a^2b^2}{(a^2 + b^2)^2} = \frac{(a^2 + b^2)^2}{(a^2 + b^2)^2} = 1$$

### Miscellaneous Exercise on Chapter 4

1. Evaluate:  $\left[ i^{18} + \left( \frac{1}{i} \right)^{25} \right]^3$ .

2. For any two complex numbers  $z_1$  and  $z_2$ , prove that  
 $\operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$

3. Reduce  $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$  to the standard form .
4. If  $x-iy = \sqrt{\frac{a-ib}{c-id}}$  prove that  $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$  .
5. If  $z_1 = 2-i, z_2 = 1+i$ , find  $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$  .
6. If  $a+ib = \frac{(x+i)^2}{2x^2+1}$ , prove that  $a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$  .
7. Let  $z_1 = 2-i, z_2 = -2+i$ . Find
- (i)  $\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ , (ii)  $\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$  .
8. Find the real numbers  $x$  and  $y$  if  $(x-iy)(3+5i)$  is the conjugate of  $-6-24i$ .
9. Find the modulus of  $\frac{1+i}{1-i} - \frac{1-i}{1+i}$  .
10. If  $(x+iy)^3 = u+iv$ , then show that  $\frac{u}{x} + \frac{v}{y} = 4(x^2-y^2)$  .
11. If  $\alpha$  and  $\beta$  are different complex numbers with  $|\beta|=1$ , then find  $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$  .
12. Find the number of non-zero integral solutions of the equation  $|1-i|^x = 2^x$  .
13. If  $(a+ib)(c+id)(e+if)(g+ih) = A+iB$ , then show that  $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$
14. If  $\left(\frac{1+i}{1-i}\right)^m = 1$ , then find the least positive integral value of  $m$ .

### Summary

- ◆ A number of the form  $a + ib$ , where  $a$  and  $b$  are real numbers, is called a *complex number*,  $a$  is called the *real part* and  $b$  is called the *imaginary part* of the complex number.
- ◆ Let  $z_1 = a + ib$  and  $z_2 = c + id$ . Then
  - (i)  $z_1 + z_2 = (a + c) + i(b + d)$
  - (ii)  $z_1 z_2 = (ac - bd) + i(ad + bc)$
- ◆ For any non-zero complex number  $z = a + ib$  ( $a \neq 0, b \neq 0$ ), there exists the complex number  $\frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}$ , denoted by  $\frac{1}{z}$  or  $z^{-1}$ , called the *multiplicative inverse* of  $z$  such that  $(a + ib) \left( \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2} \right) = 1 + i0 = 1$
- ◆ For any integer  $k$ ,  $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$
- ◆ The conjugate of the complex number  $z = a + ib$ , denoted by  $\bar{z}$ , is given by  $\bar{z} = a - ib$ .

### Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. “He mentions in his work ‘*Ganitasara Sangraha*’ as in the nature of things a negative (quantity) is not a square (quantity)”, it has, therefore, no square root”. *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. “There is no square root of a negative quantity, for it is not a square.” *Cardan* (1545) considered the problem of solving

$$x + y = 10, xy = 40.$$

He obtained  $x = 5 + \sqrt{-15}$  and  $y = 5 - \sqrt{-15}$  as the solution of it, which was discarded by him by saying that these numbers are ‘useless’. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol  $i$  for  $\sqrt{-1}$  and *W.R. Hamilton* (about 1830) regarded the complex number  $a + ib$  as an ordered pair of real numbers  $(a, b)$  thus giving it a purely mathematical definition and avoiding use of the so called ‘*imaginary numbers*’.

