

Congruency of Triangles

SAS Congruence Rule

Congruency of Triangles

Congruency of triangles helps us find solutions to many problems in real life. For example, the distance travelled by a ball in a golf course is easy to measure when the ball is on land; however, when the land is separated by a water body like a pond or any other thing, the task of measurement becomes difficult. In such cases, we use the concept of congruency.

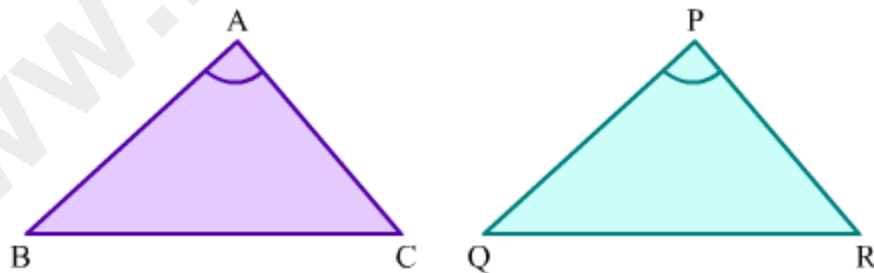
There are certain rules to check the congruency of triangles. One of these is the SAS (Side-Angle-Side) congruence rule. In this lesson, we will learn this rule and its applications.

SAS Congruence Rule

Consider a triangle two of whose sides and the included angle are known. We can check for the congruency of this triangle with respect to another triangle if we know the corresponding sides and angle of that triangle. Two triangles can, thus, be termed 'congruent' or 'incongruent' by using the SAS congruence rule. This rule states that:

If two sides of a triangle and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.

Look at the given $\triangle ABC$ and $\triangle PQR$.



Let us consider sides AB and AC and the included $\angle BAC$ in $\triangle ABC$, and the corresponding sides and angle in $\triangle PQR$, i.e., PQ , PR and $\angle QPR$.

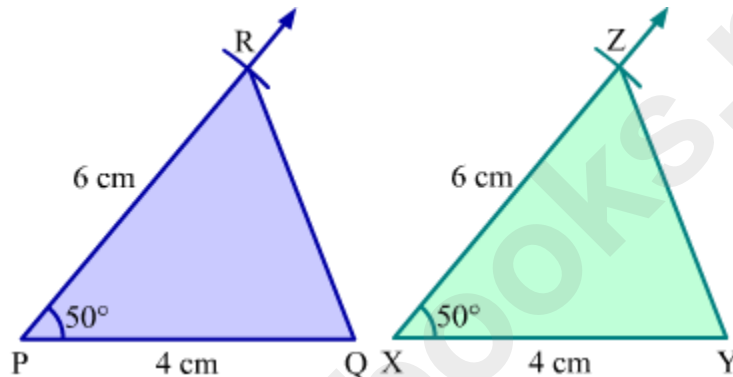
By the SAS congruence rule, the two triangles will be congruent if $AB = PQ$, $AC = PR$ and $\angle BAC = \angle QPR$.

Similarly, we can check for congruency by taking other pairs of sides and included angles in these triangles.

Verification of SAS Congruence Rule

The SAS congruence rule for triangles is taken as a postulate, so there is no proof for the same but we can verify it by doing an activity.

The steps of the activity are as follows:



- i) Construct ΔPQR such that $PQ = 4$ cm, $PR = 6$ cm and $\angle QPR = 50^\circ$.
- ii) Construct ΔXYZ with the same measures such that $XY = 4$ cm, $XZ = 6$ cm and $\angle YXZ = 50^\circ$.
- iii) Cut both the triangles along their boundaries.
- iv) Try to superpose one triangle by the other. One triangle can be placed on the other in six different ways such that vertex lie on vertex.
- v) In one of the the trials, you will get P falling over X, Q falling over Y and R falling over Z. In this case, you will see that both the triangles cover each other exactly.
- vi) Thus, under the correspondence $PQR \leftrightarrow XYZ$, the triangles are congruent.

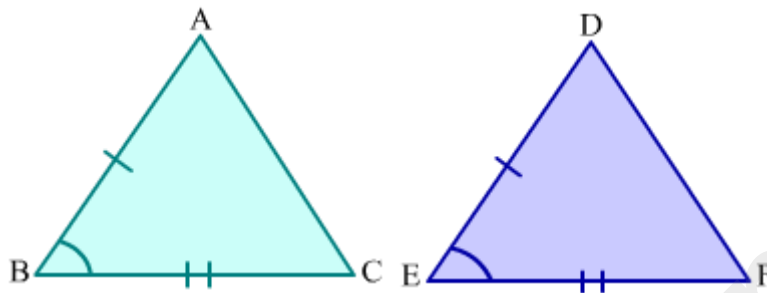
This verifies the SAS congruence rule.

Solving an Example Using the SAS Congruence Rule

CPCT

CPCT stands for 'corresponding parts of congruent triangles'. 'Corresponding parts' means corresponding sides and angles of triangles. According to CPCT, if two or more triangles are congruent to one another, then all of their corresponding parts are equal.

For example, in the given $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $\angle B = \angle E$ and $BC = EF$. So, according to the SAS congruence criterion, we have $\triangle ABC \cong \triangle DEF$.



Now, by CPCT, we can say that the remaining corresponding parts of the two congruent triangles are also equal. This means that $AC = DF$, $\angle A = \angle D$ and $\angle C = \angle F$.

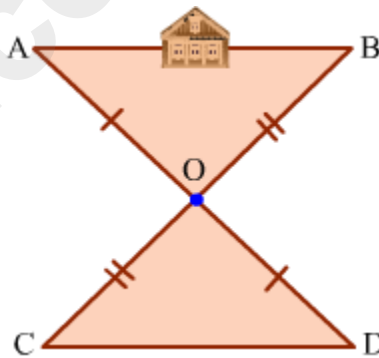
Similarly, we can apply CPCT in case of other congruent triangles.

Solved Examples

Easy

Example 1:

Observe the following figure.



Ajay wishes to determine the distance between two objects A and B, but there is a house in between. So, he devises an ingenious way to fix the problem. First, he fixes a pole at any point O so that both A and B are visible from O. He then fixes another pole at point D which is collinear to point O and object A, and is at the same distance from O as A, i.e., $DO = AO$. Similarly, he fixes a pole at point C which is collinear to point O and object B, and is at the same distance from O as B, i.e., $CO = BO$. Finally, he measures CD to find the distance between A and B. How can Ajay be sure that $CD = AB$?

Solution:

We have two triangles in the given figure, i.e., $\triangle AOB$ and $\triangle DOC$.

In these two triangles, we have:

$$AO = DO \text{ (Given)}$$

$$\angle AOB = \angle DOC \text{ (Vertically opposite angles)}$$

$$BO = CO \text{ (Given)}$$

Therefore, by the SAS congruence rule, we can say that:

$$\triangle AOB \cong \triangle DOC$$

$$\Rightarrow AB = CD \text{ (By CPCT)}$$

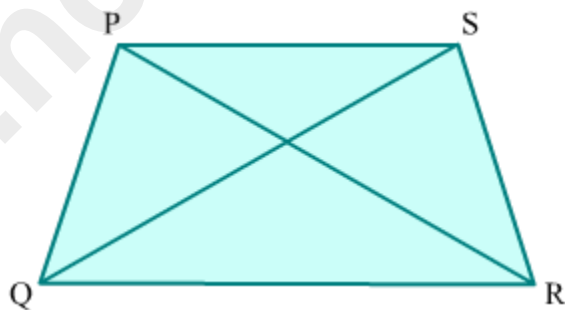
This is the reason why Ajay measures CD to find the distance between objects A and B.

Example 2:

In the given quadrilateral PQRS, PR bisects $\angle QPS$ and $PQ = PS$. Prove that:

i) $\triangle PQR \cong \triangle PSR$

ii) $QR = SR$



Solution:

i) In $\triangle PQR$ and $\triangle PSR$, we have:

$$PQ = PS \text{ (Given)}$$

$$PR = PR \text{ (Common side)}$$

$$\angle QPR = \angle SPR \text{ (because PR bisects } \angle QPS)$$

So, by the SAS congruence rule, we obtain:

$$\Delta PQR \cong \Delta PSR$$

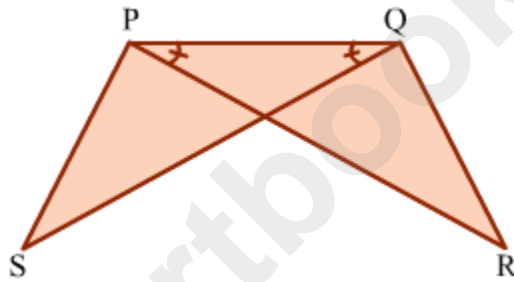
ii) We have proved that $\Delta PQR \cong \Delta PSR$.

$\therefore QR = SR$ (&because Corresponding parts of congruent triangles are equal)

Medium

Example 1:

In the shown figure, $PR = QS$ and $\angle QPR = \angle PQS$. Prove that $\Delta PQR \cong \Delta QPS$. Also, show that $PS = QR$ and $\angle QPS = \angle PQR$.



Solution:

In ΔPQR and ΔQPS , we have:

$$PR = QS \text{ (Given)}$$

$$\angle QPR = \angle PQS \text{ (Given)}$$

$$PQ = PQ \text{ (Common side)}$$

$\therefore \Delta PQR \cong \Delta QPS$ (By the SAS congruence criterion)

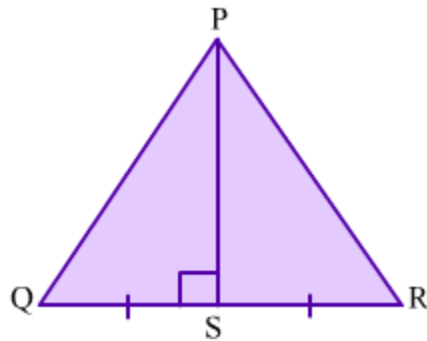
$\Rightarrow PS = QR$ and $\angle QPS = \angle PQR$ (By CPCT)

Example 2:

Prove that ΔPQR is isosceles if the altitude drawn from a vertex bisects the opposite side.

Solution:

The given figure shows the ΔPQR having PS as an altitude that bisects the opposite side QR .



In ΔPSQ and ΔPSR , we have:

$QS = SR$ (&because Altitude PS bisects QR)

$PS = PS$ (Common side)

$\angle PSQ = \angle PSR = 90^\circ$ (&because PS is the altitude to QR)

$\therefore \Delta PSQ \cong \Delta PSR$ (By the SAS congruence rule)

$\Rightarrow PQ = PR$ (By CPCT)

Therefore, ΔPQR is isosceles.

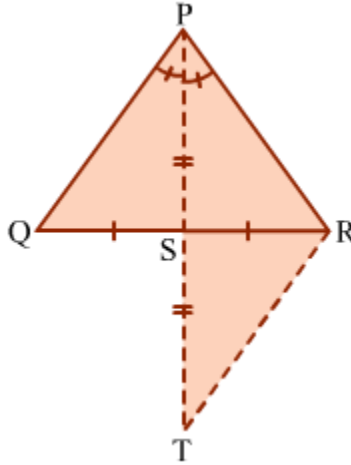
Example 3:

If the angle bisector of any angle of a triangle bisects the opposite side then show that the triangle is isosceles.

Solution:

Let ΔPQR be the given triangle and PS is the angle bisector of $\angle QPR$ such that it bisects the side QR .

Let us extend the segment PS to point T such that $PS = TS$.



In ΔPQS and ΔTRS , we have

$$QS = RS \quad (\text{Given})$$

$$\angle PSQ = \angle TSR \quad (\text{Vertically opposite angles})$$

$$PS = TS \quad (\text{By construction})$$

So, by the SAS congruence criterion, we have:

$$\Delta PQS \cong \Delta TRS$$

By CPCT, we obtain

$$PQ = TR \quad \dots(1)$$

$$\text{And } \angle QPS = \angle RTS \quad \dots(2)$$

$$\text{But } \angle QPS = \angle RPS \quad \dots(3) \quad (\text{PS bisects } \angle QPR)$$

$$\therefore \angle RTS = \angle RPS \quad [\text{From (2) and (3)}]$$

$$\Rightarrow PR = TR \quad (\text{Sides opposite to equal angles})$$

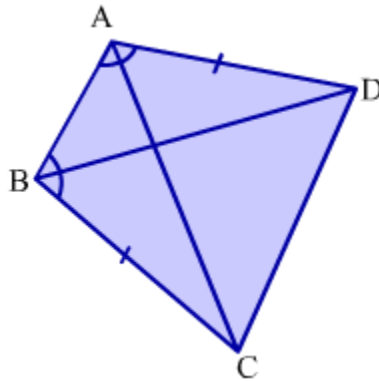
$$\therefore PQ = PR \quad [\text{From (1)}]$$

Thus, ΔPQR is an isosceles triangle.

Hard

Example 1:

ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$.



Prove that:

- i) $\triangle ABD \cong \triangle BAC$
- ii) $BD = AC$
- iii) $\angle ABD = \angle BAC$

Solution:

i) In $\triangle ABD$ and $\triangle BAC$, we have:

$AD = BC$ (Given)

$\angle DAB = \angle CBA$ (Given)

$AB = BA$ (Common side)

So, by the SAS congruence criterion, we have:

$\triangle ABD \cong \triangle BAC$

ii) We have proved that $\triangle ABD \cong \triangle BAC$.

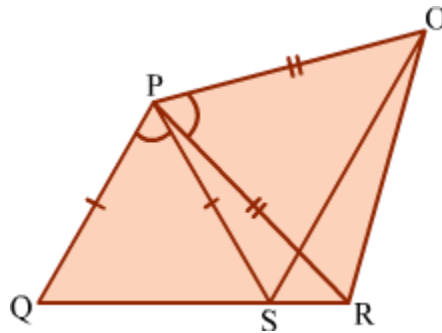
$\therefore BD = AC$ (By CPCT)

iii) Since $\triangle ABD \cong \triangle BAC$, we have:

$\angle ABD = \angle BAC$ (By CPCT)

Example 2:

In the given figure, $PR = PO$, $PQ = PS$ and $\angle QPS = \angle OPR$. Show that $QR = SO$.



Solution:

It is given that $\angle QPS = \angle OPR$.

$$\therefore \angle QPS + \angle SPR = \angle OPR + \angle SPR$$

$$\Rightarrow \angle QPR = \angle SPO \dots (1)$$

In $\triangle QPR$ and $\triangle SPO$, we have:

$$PQ = PS \quad (\text{Given})$$

$$\angle QPR = \angle SPO \quad (\text{From equation 1})$$

$$PR = PO \quad (\text{Given})$$

So, by the SAS congruence rule, we have:

$$\triangle QPR \cong \triangle SPO$$

$$\Rightarrow QR = SO \quad (\text{By CPCT})$$

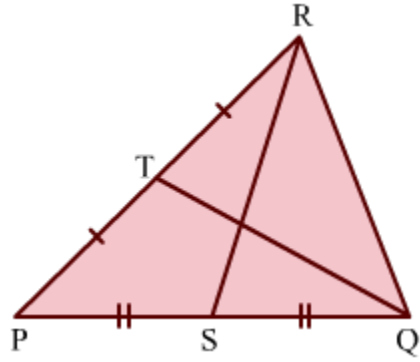
Example 3:

In an isosceles triangle, prove that the medians on the equal sides are equal.

Solution:

Let $\triangle PQR$ be an isosceles triangle such that $PQ = PR$.

Also, let RS and QT be the medians to the sides PQ and PR respectively.



In $\triangle PQR$, we have

$$PS = SQ = \frac{1}{2}PQ \quad (\text{RS is the median})$$

$$\text{And } PT = TR = \frac{1}{2}PR \quad (\text{QT is the median})$$

But $PQ = PR$

$$\therefore PS = SQ = PT = TR \quad \dots(1)$$

In $\triangle PRS$ and $\triangle PQT$, we have

$$PQ = PR \quad (\text{Given})$$

$$\angle RPS = \angle QPT \quad (\text{Common angle})$$

$$PS = PT \quad [\text{From (1)}]$$

So, by the SAS congruence rule, we have:

$$\triangle PRS \cong \triangle PQT$$

$$\therefore RS = QT \quad (\text{By CPCT})$$

Thus, the medians on the equal sides of an isosceles triangle are equal.

Proving Theorem of Right Angled Triangle

There is a theorem of right angled triangles which states that:

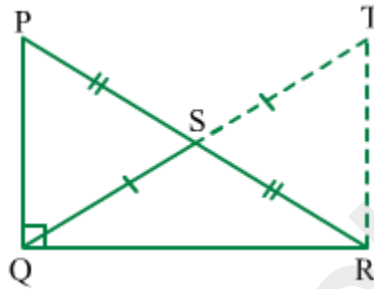
In a right angled triangle, median drawn to the hypotenuse from the opposite vertex is equal to the half of the hypotenuse.

Let us prove the theorem.

Given: Right angled ΔPQR , $\angle PQR = 90^\circ$ and median QS to hypotenuse PR .

To prove: $QS = \frac{1}{2}PR$

Construction: Extend QS to T such that $QS = ST$ and join T to R .



Proof:

In ΔPSQ and ΔRST , we have

$QS = TS$ (By construction)

$\angle PSQ = \angle RST$ (Vertically opposite angles)

$PS = RS$ (Given)

So, by SAS congruence criterion, we have

$\Delta PSQ \cong \Delta RST$

$\therefore PQ = RT$...(1) (By CPCT)

And $\angle QPS = \angle TRS$ (By CPCT)

Thus, $PQ \parallel RT$ ($\angle QPS$ and $\angle TRS$ alternate interior angles formed by transversal PR)

Now, QR is also a transversal to parallel line segments PQ and RT .

$\angle PQR + \angle TRQ = 180^\circ$ (Sum of interior angles on the same side of transversal)

But $\angle PQR = 90^\circ$

$$\therefore \angle TRQ = 90^\circ \quad (2)$$

In ΔPQR and ΔTRQ , we have

$$PQ = RT \quad [\text{From (1)}]$$

$$\angle PQR = \angle TRQ = 90^\circ$$

$$QR = QR \quad (\text{Common side})$$

So, by SAS congruence criterion, we have

$$\Delta PQR \cong \Delta TRQ$$

$$\therefore PR = QT \quad (\text{By CPCT})$$

$$\Rightarrow \frac{1}{2}PR = \frac{1}{2}QT$$

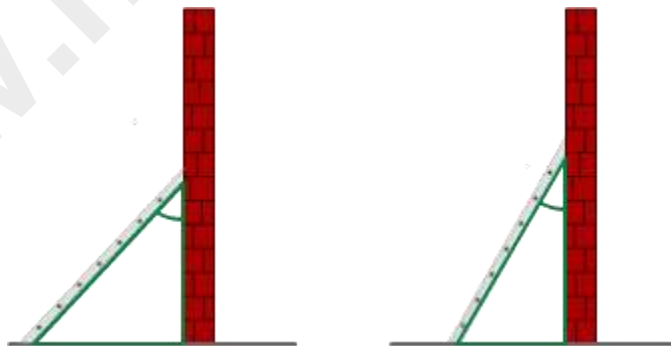
$$\text{But } \frac{1}{2}QT = QS$$

$$\therefore QS = \frac{1}{2}PR$$

Hence, proved.

ASA Congruence Rule

Look at the given figure.



Observe how the ladder, the wall and the horizontal together make triangles in the figure. It can be seen that the angle marked between the ladder and the wall on the left is greater than the same angle marked on the right. Clearly, the triangles are not congruent although it is the same ladder on both sides.

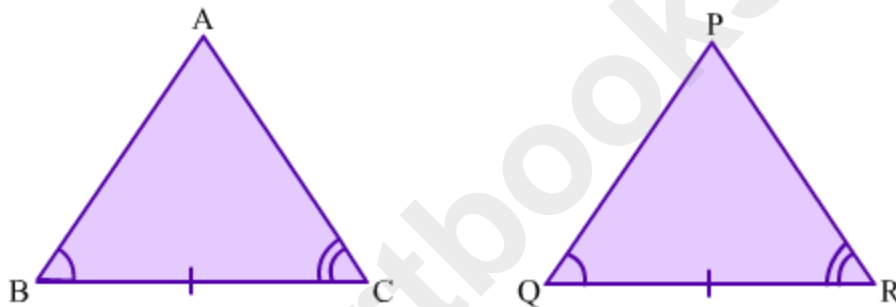
Both angles and sides play an important role in determining the congruency of triangles. In this lesson, we will discuss the ASA (Angle-Side-Angle) congruence rule and solve some problems based on it.

ASA Congruence Rule

The ASA congruence rule for triangles states that:

If two angles of a triangle and the side between them are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

Consider the given $\triangle ABC$ and $\triangle PQR$.



Observe how corresponding components of the two triangles are marked.

Now, by the ASA congruence rule, the two triangles will be congruent if these corresponding components are equal, i.e., if $\angle ABC = \angle PQR$, $BC = QR$ and $\angle ACB = \angle PRQ$, then $\triangle ABC \cong \triangle PQR$.

Note that, under the above condition of congruence, we cannot write $\triangle ABC \cong \triangle QRP$. The order of the vertices matters in any congruency.

Did You Know?

A bright meteor was seen in the sky above Greenland on December 9, 1997. In an attempt to find the fragments of the meteorite, scientists collected data from eyewitnesses who observed the meteor passing through the sky. As is shown in the figure below, the scientists considered sightlines of observers in different towns.

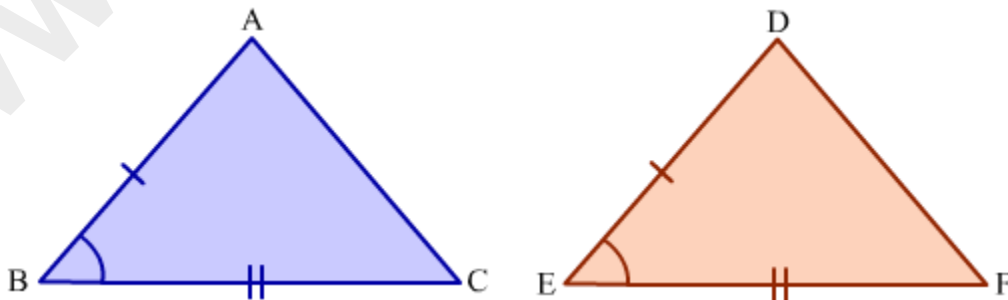


One such sightline was of observers in the town called Paamiut (Point P). Another was of observers in the town called Narsarsuaq (Point N). Using the ASA congruence rule, the scientists were able to gather enough information to successfully locate the fragments of the meteorite (Point M).

Proof of the ASA Congruence Rule: Case 1

Let us consider $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$. By the ASA congruence rule, $\triangle ABC$ and $\triangle DEF$ are congruent. By CPCT, we have $AB = DE$.

Case 1: Let us prove $\triangle ABC \cong \triangle DEF$ by taking $AB = DE$.



In this case, we have

$AB = DE$ (Given)

$$\angle ABC = \angle DEF \text{ (Given)}$$

$$BC = EF \text{ (Given)}$$

So, by the SAS congruence rule, we have:

$$\triangle ABC \cong \triangle DEF$$

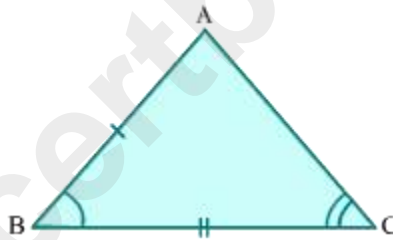
Proof of the ASA Congruence Rule: Case 2

Let us consider $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$. By the ASA congruence rule, $\triangle ABC$ and $\triangle DEF$ are congruent. By CPCT, we have $AB = DE$. Let us assume $AB \neq DE$.

Case 2: Let us prove $\triangle ABC \cong \triangle DEF$ by taking $AB < DE$.

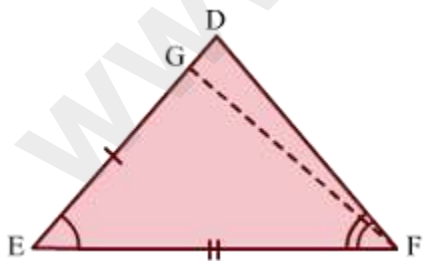
Construction: Mark a point G on DE such that $GE = AB$. Join G to F.

Now, in $\triangle ABC$ and $\triangle GEF$, we have:



$$AB = GE \text{ (By construction)}$$

$$\angle ABC = \angle GEF \text{ } (\because \angle ABC = \angle DEF \text{ and } \angle DEF = \angle GEF)$$



$$BC = EF \text{ (Given)}$$

So, by the SAS congruence rule, we obtain:

$$\triangle ABC \cong \triangle GEF$$

$\Rightarrow \angle ACB = \angle GFE$ (By CPCT)

But $\angle ACB = \angle DFE$ (Given)

$\therefore \angle GFE = \angle DFE$

This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and $\triangle GEF$ must be $\triangle DEF$.

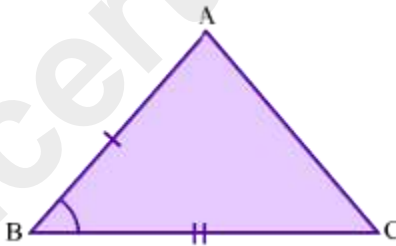
$\therefore \triangle ABC \cong \triangle DEF$

Proof of the ASA Congruence Rule: Case 3

Let us consider $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$. By the ASA congruence rule, $\triangle ABC$ and $\triangle DEF$ are congruent. By CPCT, we have $AB = DE$. Let us assume $AB \neq DE$.

Case 3: Let us prove $\triangle ABC \cong \triangle DEF$ by taking $AB > DE$.

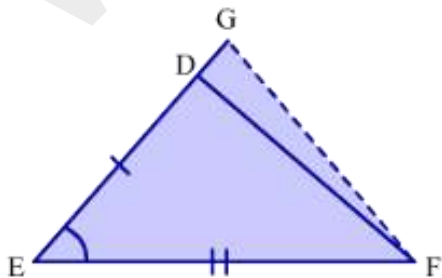
Construction: Extend ED to point G such that $GE = AB$. Join G to F.



Now, in $\triangle ABC$ and $\triangle GEF$, we have:

$AB = GE$ (By construction)

$\angle ABC = \angle DEF$ ($\because \angle ABC = \angle DEF$ and $\angle DEF = \angle GEF$)



$BC = EF$ (Given)

So, by the SAS congruence rule, we obtain:

$\triangle ABC \cong \triangle GEF$

$\Rightarrow \angle ACB = \angle GFE$ (By CPCT)

But $\angle ACB = \angle DFE$ (Given)

$\therefore \angle GFE = \angle DFE$

This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and $\triangle GEF$ must be $\triangle DEF$.

$\therefore \triangle ABC \cong \triangle DEF$

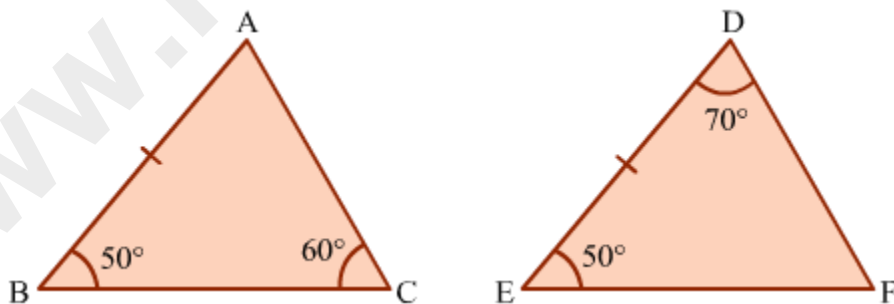
Applying the ASA Congruence Rule

Solved Examples

Easy

Example 1:

Check whether the given triangles are congruent or not.



Solution:

In $\triangle ABC$, we have:

$\angle ABC + \angle BCA + \angle BAC = 180^\circ$ (By the angle sum property)

$\Rightarrow 50^\circ + 60^\circ + \angle BAC = 180^\circ$

$$\Rightarrow 110^\circ + \angle BAC = 180^\circ$$

$$\Rightarrow \angle BAC = 180^\circ - 110^\circ$$

$$\Rightarrow \angle BAC = 70^\circ$$

In $\triangle ABC$ and $\triangle DEF$, we have:

$$\angle BAC = \angle EDF = 70^\circ$$

$$AB = DE \text{ (Given)}$$

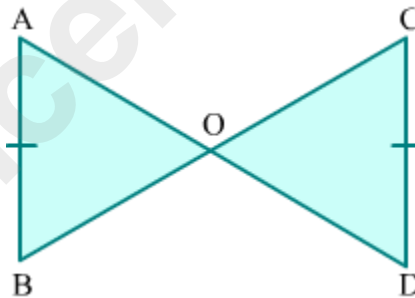
$$\angle ABC = \angle DEF = 50^\circ$$

Therefore, by the ASA congruence rule, we have:

$$\triangle ABC \cong \triangle DEF$$

Example 2:

In the given figure, AB and CD are two equal and parallel lines. Prove that $\triangle ABO \cong \triangle CDO$.



Solution:

It is given that $AB \parallel CD$. AD and BC are transversals lying on lines AB and CD .

So, by the alternate angles axiom, we obtain:

$$\angle OAB = \angle ODC \dots (1)$$

$$\angle OBA = \angle OCD \dots (2)$$

In $\triangle ABO$ and $\triangle CDO$, we have:

$$\angle OAB = \angle ODC \text{ (By equation 1)}$$

$$AB = CD \text{ (Given)}$$

$$\angle OBA = \angle OCD \text{ (By equation 2)}$$

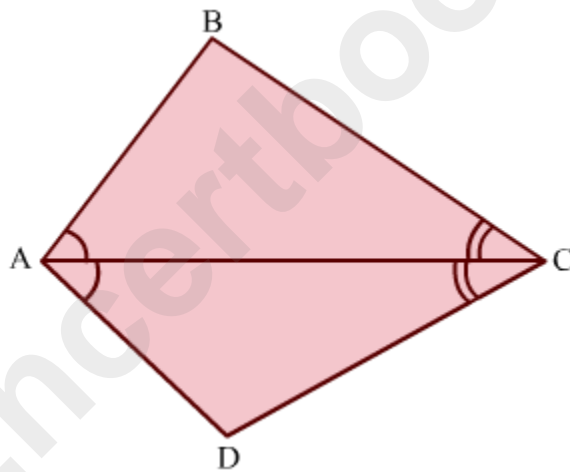
Thus, by the ASA congruence rule, we obtain:

$$\triangle ABO \cong \triangle CDO$$

Medium

Example 1:

In the given quadrilateral ABCD, diagonal AC bisects $\angle BAD$ and $\angle BCD$. Prove that $AB = AD$ and $CB = CD$.



Solution:

Since diagonal AC bisects $\angle BAD$ and $\angle CAD$, we have:

$$\angle BAC = \angle DAC \text{ and } \angle BCA = \angle DCA$$

In $\triangle ACB$ and $\triangle ACD$, we have:

$$\angle BAC = \angle DAC \text{ (Given)}$$

$$\angle BCA = \angle DCA \text{ (Given)}$$

$$AC = AC \text{ (Common side)}$$

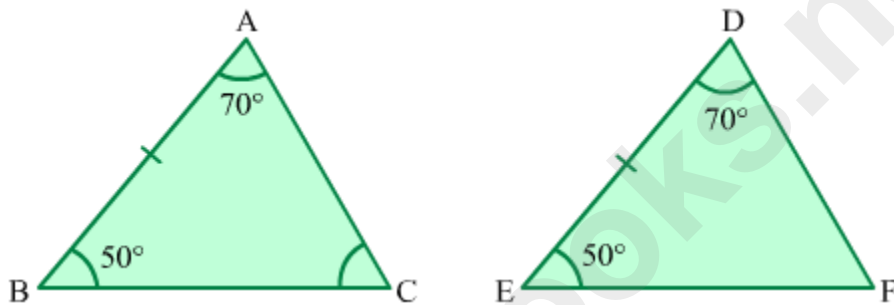
So, by the ASA congruence rule, we have:

$$\triangle ACB \cong \triangle ACD$$

$\Rightarrow AB = AD$ and $CB = CD$ (By CPCT)

Example 2:

Consider the two triangular parks ABC and DEF shown below.



Tina jogs around park ABC and Aliya jogs around park DEF daily. Paths AB and DE are equal in length. If both girls jog an equal number of rounds daily, then check whether or not they cover the same distance while jogging?

Solution:

In $\triangle ABC$ and $\triangle DEF$, we have:

$$\angle BAC = \angle EDF = 70^\circ \text{ (Given)}$$

$$AB = DE \text{ (Given)}$$

$$\angle ABC = \angle DEF = 50^\circ \text{ (Given)}$$

Therefore, by the ASA congruency rule, we obtain:

$$\triangle ABC \cong \triangle DEF$$

$\Rightarrow AC = DF$ and $BC = EF$ (By CPCT)

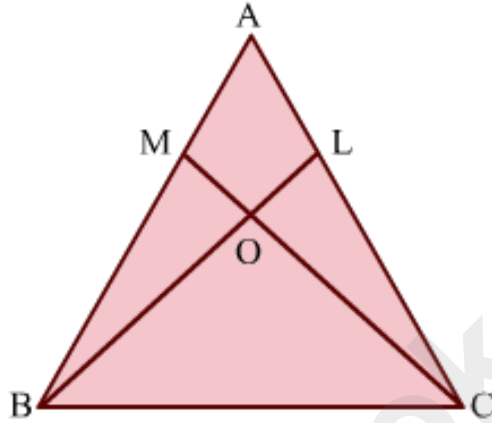
$$\therefore AB + BC + CA = DE + EF + FD$$

Hence, both Tina and Aliya cover the same distance daily while jogging.

Hard

Example 1:

The given $\triangle ABC$ is isosceles with $AB = AC$. $\angle LOC = 2\angle OBC$ and $\angle MOB = 2\angle OCB$. Prove that $\triangle BCM \cong \triangle CBL$.



Solution:

It is given that:

$$\angle LOC = 2\angle OBC \dots (1)$$

$$\angle MOB = 2\angle OCB \dots (2)$$

Now, $\angle LOC = \angle MOB$ (Vertically opposite angles)

Using equations (1) and (2), we obtain:

$$\angle OCB = \angle OBC$$

$$\Rightarrow \angle MCB = \angle LBC \dots (3)$$

In $\triangle BCM$ and $\triangle CBL$, we have:

$$\angle MBC = \angle LCB \quad (\because \triangle ABC \text{ is isosceles with } AB = AC)$$

$$BC = CB \quad (\text{Common side})$$

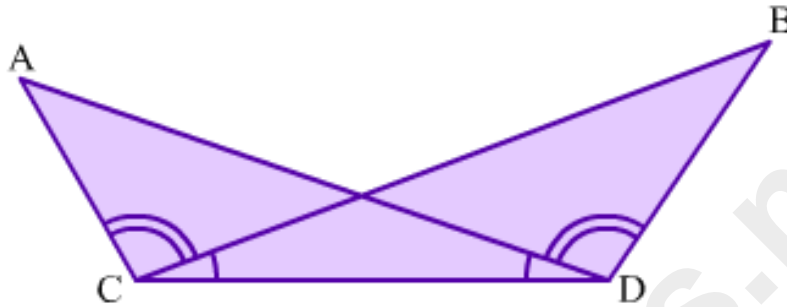
$$\angle MCB = \angle LBC \quad (\text{Using equation 3})$$

Thus, by the ASA congruence rule, we obtain:

$$\triangle BCM \cong \triangle CBL$$

Example 2:

In the given figure, $\angle BCD = \angle ADC$ and $\angle ACB = \angle BDA$. Prove that $AD = BC$ and $\angle CAD = \angle DBC$.

**Solution:**

It is given that:

$$\angle BCD = \angle ADC \dots (1)$$

$$\angle ACB = \angle BDA \dots (2)$$

On adding equations (1) and (2), we get:

$$\angle BCD + \angle ACB = \angle ADC + \angle BDA$$

$$\Rightarrow \angle ACD = \angle BDC \dots (3)$$

In $\triangle ACD$ and $\triangle BDC$, we have:

$$\angle ADC = \angle BCD (\text{Given})$$

$$CD = DC \text{ (Common side)}$$

$$\angle ACD = \angle BDC \text{ (By equation 3)}$$

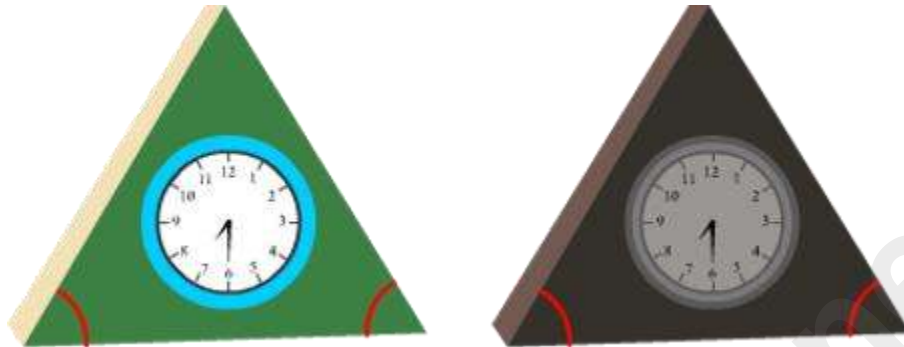
So, by the ASA congruence rule, we have:

$$\triangle ACD \cong \triangle BDC$$

$$\Rightarrow AD = BC \text{ and } \angle CAD = \angle DBC \text{ (By CPCT)}$$

AAS Congruence Rule

Consider the following triangular clocks.



In each clock, two angles on the base are marked. We can check the congruency of the clocks if we know the marked angles and the included side (i.e., the base) by using the ASA congruence rule.

Now, can we check the congruency of these clocks if instead of the base we are given any of the other two sides? Yes, we can do so and this is why we have the AAS (Angle-Angle-Side) congruence rule. In this lesson, we will discuss this rule and solve some examples related to it.

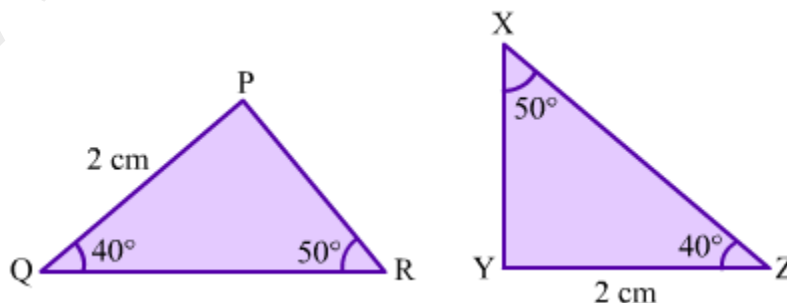
AAS Congruence Rule

The AAS congruence rule for triangles states that:

If any two angles and one side of a triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

This is nothing but the generalized form of the ASA congruence rule.

Consider the given $\triangle PQR$ and $\triangle XYZ$.



In each triangle, the given side is not between the given angles. So, we cannot apply the ASA congruence rule here. Instead, we will use the AAS congruence rule.

In $\triangle PQR$ and $\triangle YZX$, we have:

$$PQ = YZ = 2 \text{ cm}$$

$$\angle PQR = \angle YZX = 40^\circ$$

$$\angle PRQ = \angle YXZ = 50^\circ$$

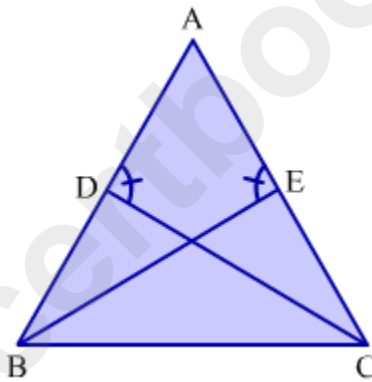
Thus, by using the AAS congruence criterion, we obtain: $\Delta PQR \cong \Delta YZX$

Solved Examples

Easy

Example 1:

The given ΔABC is isosceles with $AB = AC$ and $\angle ADC = \angle AEB$. Prove that $\Delta ABE \cong \Delta ACD$.



Solution:

In ΔABE and ΔACD , we have:

$$AB = AC \text{ (Given)}$$

$$\angle AEB = \angle ADC \text{ (Given)}$$

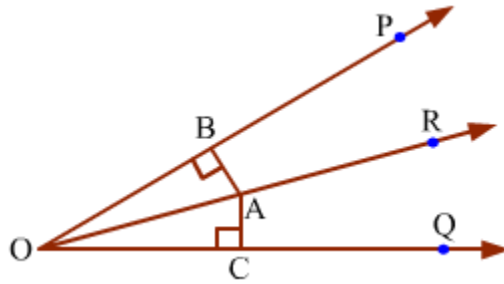
$$\angle BAE = \angle CAD \text{ (Common angle)}$$

Thus, by the AAS congruence rule, we obtain:

$$\Delta ABE \cong \Delta ACD$$

Example 2:

In the given figure, OR bisects $\angle POQ$ and A is any point on OR. AB and AC are the perpendiculars drawn from A to the arms OP and OQ respectively. Prove that $\triangle AOB \cong \triangle AOC$.



Solution:

It is given that OR bisects $\angle POQ$.

$$\therefore \angle POR = \angle QOR$$

$$\Rightarrow \angle BOA = \angle COA \dots (1)$$

In $\triangle AOB$ and $\triangle AOC$, we have:

$$\angle ABO = \angle ACO = 90^\circ \quad (\because AB \text{ and } AC \text{ are perpendiculars})$$

$$\angle BOA = \angle COA \quad (\text{By equation 1})$$

$$AO = AO \quad (\text{Common side})$$

Thus, by the AAS congruence rule, we obtain:

$$\triangle AOB \cong \triangle AOC$$

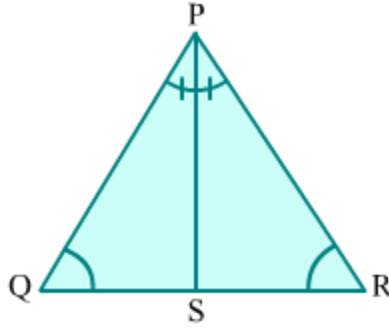
Medium

Example 1:

If two angles of a triangle are equal, then prove that the sides opposite them are also equal.

Solution:

Consider a $\triangle PQR$ with $\angle PQR = \angle PRQ$.



We have to prove that $PQ = PR$.

Construction: Draw the bisector of $\angle QPR$ and let it meet side QR at point S .

In $\triangle PSQ$ and $\triangle PSR$, we have:

$$\angle PQS = \angle PRS \text{ (Given)}$$

$$PS = PS \text{ (Common side)}$$

$$\angle QPS = \angle RPS \text{ (}\because PS \text{ bisects } \angle QPR)$$

So, by the AAS congruence rule, we obtain:

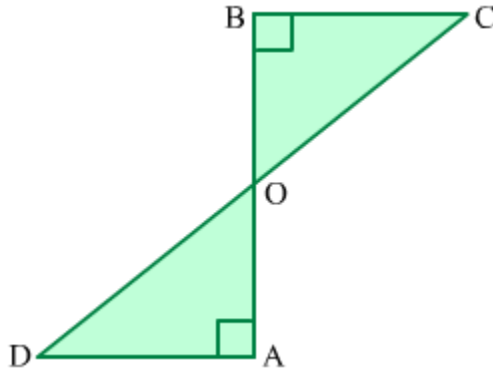
$$\triangle PSQ \cong \triangle PSR$$

$$\Rightarrow PQ = PR \text{ (By CPCT)}$$

Hence, we have proved that the sides opposite equal angles of a triangle are also equal.

Example 2:

DA and CB are equal perpendiculars to a line segment AB . Show that line segment CD bisects AB at point O .



Solution:

In $\triangle DAO$ and $\triangle CBO$, we have:

$\angle AOD = \angle BOC$ (Vertically opposite angles)

$\angle OAD = \angle OBC = 90^\circ$ (\because DA and CB are perpendiculars)

DA = CB (Given)

So, by the AAS congruence rule, we obtain:

$\triangle DAO \cong \triangle CBO$

$\Rightarrow AO = BO$ (By CPCT)

Now, $AO + BO = AB$

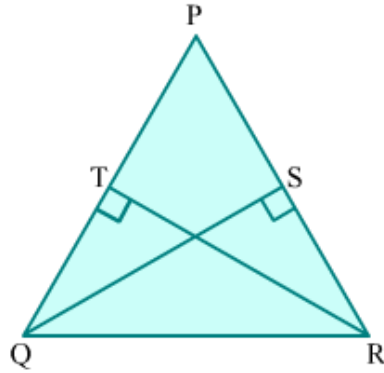
Thus, line segment CD bisects line segment AB at point O.

Example 3:

If two altitudes of a triangle are equal then show that the triangle is isosceles.

Solution:

Let us consider $\triangle PQR$ such that RT and QS are the equal altitudes drawn to the sides PQ and PR respectively.



In ΔPTR and ΔPSQ , we have

$$\angle PTR = \angle PSQ = 90^\circ$$

$$\angle TPR = \angle SPQ \quad (\text{Common angle})$$

$$RT = QS \quad (\text{Given})$$

So, by the AAS congruence rule, we obtain:

$$\Delta PTR \cong \Delta PSQ$$

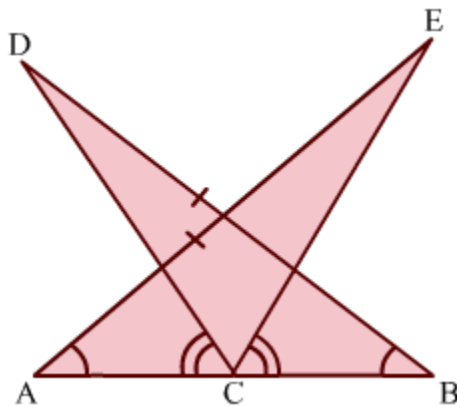
$$\therefore PR = PQ \quad (\text{By CPCT})$$

Thus, ΔPQR is an isosceles triangle.

Hard

Example 1:

In the given figure, $\angle DBC = \angle EAC$, $\angle DCA = \angle ECB$ and $BD = AE$. Prove that $BC = AC$.



Solution:

It is given that $\angle DCA = \angle ECB$.

On adding $\angle ECD$ to both sides of the above equation, we get:

$$\angle DCA + \angle ECD = \angle ECB + \angle ECD$$

$$\Rightarrow \angle ECA = \angle DCB \dots (1)$$

In $\triangle BDC$ and $\triangle AEC$, we have:

$$\angle DCB = \angle ECA \text{ (From equation 1)}$$

$$\angle DBC = \angle EAC \text{ (Given)}$$

$$BD = AE \text{ (Given)}$$

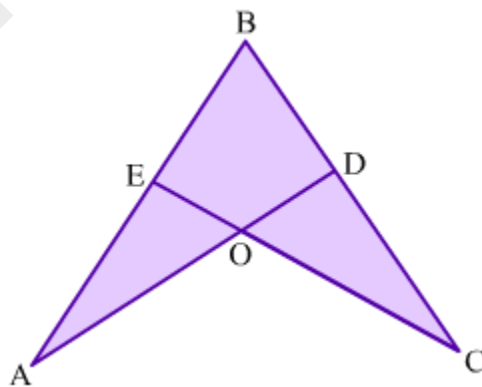
So, by the AAS congruence rule, we have:

$$\triangle BDC \cong \triangle AEC$$

$$\Rightarrow BC = AC \text{ (By CPCT)}$$

Example 2:

In the figure, $\angle BAD = \angle BCE$ and $AB = CB$. Prove that $\triangle ABD \cong \triangle CBE$.



Solution:

In $\triangle AOE$ and $\triangle COD$, we have:

$$\angle EAO = \angle DCO \text{ (Given)}$$

$\angle AOE = \angle COD$ (Vertically opposite angles)

$\therefore \angle AEO = \angle CDO$... (1) [By the angle sum property]

Now,

$\angle AEO + \angle OEB = 180^\circ$ (Linear pair)

$\angle CDO + \angle ODB = 180^\circ$ (Linear pair)

$\therefore \angle AEO + \angle OEB = \angle CDO + \angle ODB$

$\Rightarrow \angle AEO + \angle OEB = \angle AEO + \angle ODB$ (Using equation (1))

$\Rightarrow \angle OEB = \angle ODB$

$\Rightarrow \angle CEB = \angle ADB$... (2)

In $\triangle ABD$ and $\triangle CBE$, we have:

$\angle BAD = \angle BCE$ (Given)

$\angle ADB = \angle CEB$ (From equation 2)

$AB = CB$ (Given)

So, by the AAS congruence rule, we have:

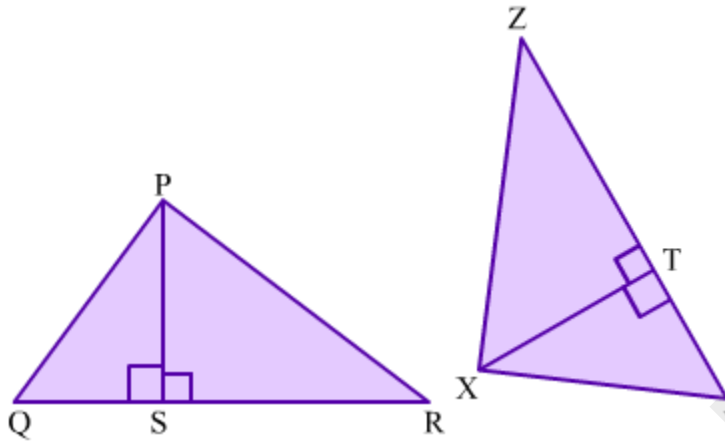
$\triangle ABD \cong \triangle CBE$

Example 3:

Prove that if two triangles are congruent then their corresponding altitudes are equal.

Solution:

Let PS and XT be corresponding altitudes of congruent triangles $\triangle PQR$ and $\triangle XYZ$.



We have

$$\Delta PQR \cong \Delta XYZ$$

$$\therefore PQ = XY \quad \dots(1) \quad (\text{By CPCT})$$

$$\angle PQR = \angle XYZ \quad \dots(2) \quad (\text{By CPCT})$$

In ΔPQS and ΔXYT , we have

$$\angle PSQ = \angle XTY = 90^\circ$$

$$\angle PQR = \angle XYZ \quad [\text{From (2)}]$$

$$PQ = XY \quad [\text{From (1)}]$$

So, by the AAS congruence rule, we have:

$$\Delta PQS \cong \Delta XYT$$

$$\therefore PS = XT \quad (\text{By CPCT})$$

Thus, the corresponding altitudes of congruent triangles are equal.

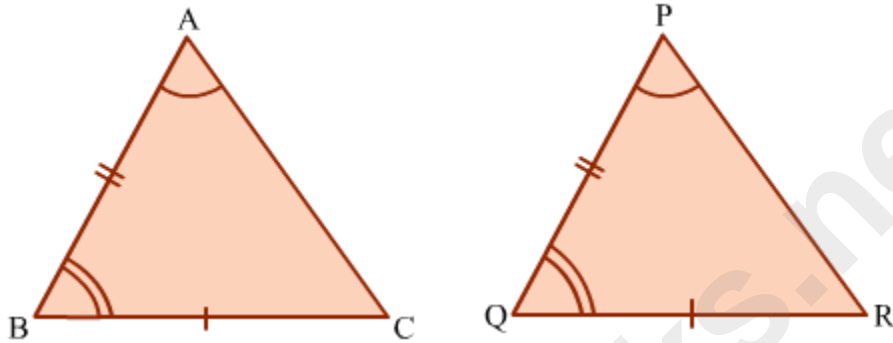
Difference between AAS and ASA

The AAS and ASA criteria are both used for checking or proving congruency of triangles, but there is a subtle difference between the two.

- The ASA rule is applied when two interior angles of a triangle and the side between them are considered.

- The AAS rule is applied when two interior angles of a triangle and any side other than the one between them are considered.

Take, for example, the given $\triangle ABC$ and $\triangle PQR$.



To prove $\triangle ABC \cong \triangle PQR$ by the ASA rule, we need:

$\angle BAC = \angle QPR$, $\angle ABC = \angle PQR$ and $AB = PQ$

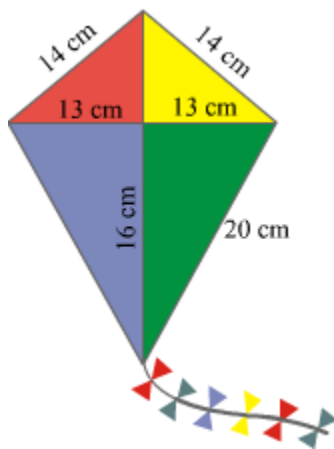
To prove $\triangle ABC \cong \triangle PQR$ by the AAS rule, we need:

$\angle BAC = \angle QPR$, $\angle ABC = \angle PQR$ and $BC = QR$

SSS Congruence Rule

Relation between the Congruency of Triangles and Their Sides

Consider the kite shown below.



It can be seen that the red and yellow coloured triangles have equal sides. On the basis of this information, can we say that the two triangles are congruent? Or, to rephrase the

question, do the sides of triangles determine the congruency of the triangles? Yes, they do, and this is why we have the SSS (Side-Side-Side) congruence rule.

In this lesson, we will discuss the SSS congruence rule and its proof. We will also crack some problems based on it.

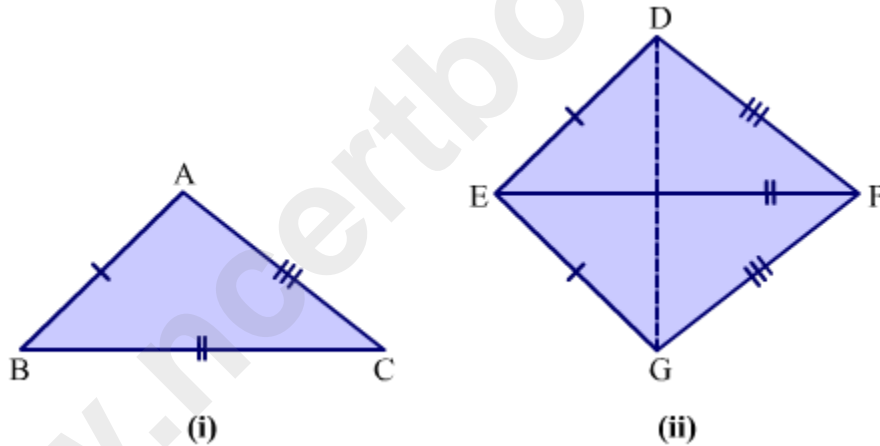
Proving the SSS Congruence Rule

Statement: Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

Given: $\triangle ABC$ and $\triangle DEF$ such that $AB = DE$, $BC = EF$ and $AC = DF$

To prove: $\triangle ABC \cong \triangle DEF$

Construction: Suppose BC and EF are the longest sides of the two triangles. Draw EG such that $\angle GEF = \angle ABC$ and $GE = AB$. Join point G to points F and D .



Proof: In $\triangle ABC$ and $\triangle GEF$, we have:

$BC = EF$ (Given)

$AB = GE$ (By construction)

$\angle ABC = \angle GEF$ (By construction)

So, by the SAS congruence rule, we have:

$\triangle ABC \cong \triangle GEF$

$\Rightarrow \angle BAC = \angle EGF$ and $AC = GF$ (By CPCT)

Now, $AB = DE$ and $AB = GE$

$$\Rightarrow DE = GE \dots (1)$$

Similarly, $AC = DF$ and $AC = GF$

$$\Rightarrow DF = GF \dots (2)$$

In $\triangle DEG$, we have:

$$DE = GE \text{ (From equation 1)}$$

$$\Rightarrow \angle EDG = \angle EGD \dots (3)$$

In $\triangle DFG$, we have:

$$DF = GF \text{ (From equation 2)}$$

$$\Rightarrow \angle FDG = \angle FGD \dots (4)$$

On adding equations 3 and 4, we get:

$$\angle EDG + \angle FDG = \angle EGD + \angle FGD$$

$$\Rightarrow \angle EDF = \angle EGF$$

We know that $\angle EGF = \angle BAC$ (Proved above)

$$\therefore \angle BAC = \angle EDF \dots (5)$$

Thus, in $\triangle ABC$ and $\triangle DEF$, we have:

$$AB = DE \text{ (Given)}$$

$$\angle BAC = \angle EDF \text{ (From equation 5)}$$

$$AC = DF \text{ (Given)}$$

So, by the SAS congruence rule, we obtain:

$$\triangle ABC \cong \triangle DEF$$

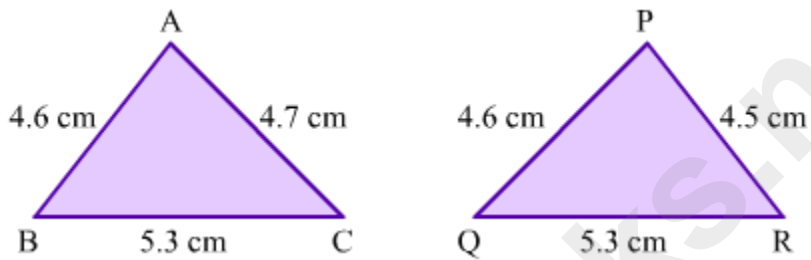
Hence, the SSS congruence rule holds true.

Solved Examples

Easy

Example 1:

Are the following triangles congruent?



Solution:

In $\triangle ABC$ and $\triangle PQR$, we have:

$$AB = PQ = 4.6 \text{ cm}$$

$$BC = QR = 5.3 \text{ cm}$$

$$\text{But } AC \neq PR$$

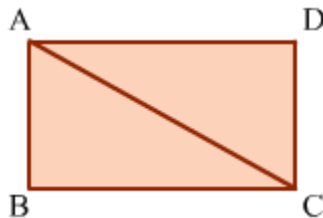
Therefore, $\triangle ABC$ and $\triangle PQR$ are not congruent.

Example 2:

ABCD is a rectangle with AC as one of its diagonals. Prove that the triangles formed on the two sides of diagonal AC are congruent.

Solution:

The required rectangle ABCD with AC as its diagonal can be drawn as is shown.



In $\triangle ABC$ and $\triangle CDA$, we have:

$AB = CD$ (\because Opposite sides of a rectangle are equal)

$BC = DA$ (\because Opposite sides of a rectangle are equal)

$CA = AC$ (Common side)

Therefore, by the SSS congruence rule, we have:

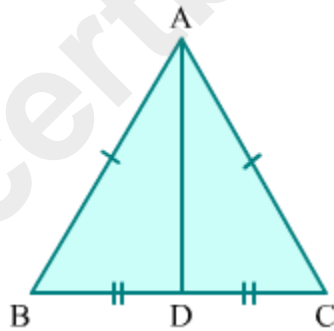
$\triangle ABC \cong \triangle CDA$

Thus, the triangles formed on the two sides of diagonal AC are congruent.

Medium

Example 1:

The given $\triangle ABC$ is isosceles with $AB = AC$. AD is a median of the triangle. Prove that AD is perpendicular to BC.



Solution:

In $\triangle ABD$ and $\triangle ACD$, we have:

$AB = AC$ (Given)

$BD = DC$ (\because D is the midpoint of BC)

$AD = AD$ (Common side)

Therefore, by the SSS congruence rule, we obtain:

$\triangle ABD \cong \triangle ACD$

$\Rightarrow \angle ADB = \angle ADC$ (By CPCT)

Also, $\angle ADB$ and $\angle ADC$ form a linear pair.

So, $\angle ADB + \angle ADC = 180^\circ$

$\Rightarrow \angle ADB + \angle ADB = 180^\circ$ ($\because \angle ADB = \angle ADC$)

$\Rightarrow 2\angle ADB = 180^\circ$

$\Rightarrow \angle ADB = 90^\circ$

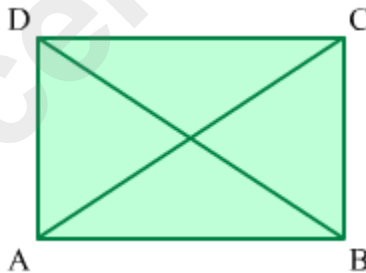
Thus, $\angle ADB = \angle ADC = 90^\circ$, which means that AD is perpendicular to BC.

Example 2:

ABCD is a parallelogram. If the diagonals of ABCD are equal, then find the measure of $\angle ABC$.

Solution:

The given parallelogram ABCD with equal diagonals AC and BD is shown below.



In parallelogram ABCD, we have:

$AB = CD$ and $AD = BC$ (\because Opposite sides of a parallelogram are equal)

In $\triangle ADB$ and $\triangle BCA$, we have:

$AD = BC$ (Proved above)

$BD = AC$ (Given)

$BA = AB$ (Common side)

So, by the SSS congruence rule, we have:

$$\triangle ADB \cong \triangle BCA$$

$$\Rightarrow \angle BAD = \angle ABC \dots (1) \text{ [By CPCT]}$$

Now, AD is parallel to BC and the transversal AB intersects them at A and B respectively.

We know that the sum of the interior angles on the same side of a transversal is supplementary.

$$\therefore \angle BAD + \angle ABC = 180^\circ$$

$$\Rightarrow \angle ABC + \angle ABC = 180^\circ \text{ (By equation 1)}$$

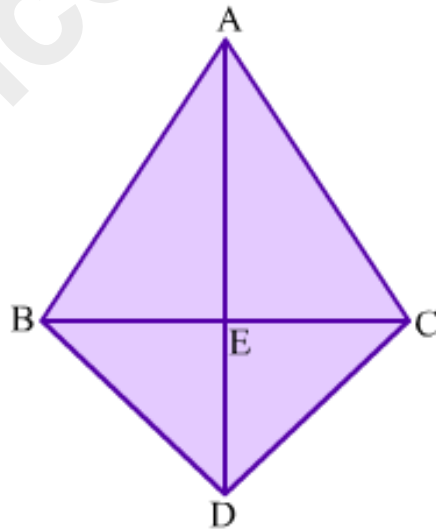
$$\Rightarrow 2\angle ABC = 180^\circ$$

$$\Rightarrow \angle ABC = 90^\circ$$

Hard

Example 1:

In the given figure, $\triangle ABC$ and $\triangle DBC$ are isosceles with $AB = AC$ and $DB = DC$. Prove that AD is the perpendicular bisector of BC.



Solution:

In $\triangle ABD$ and $\triangle ACD$, we have:

$AB = AC$ (Given)

$DB = DC$ (Given)

$AD = AD$ (Common side)

So, by the SSS congruence rule, we have:

$\triangle ABD \cong \triangle ACD$

$\Rightarrow \angle BAE = \angle CAE \dots (1)$ [By CPCT]

In $\triangle BAE$ and $\triangle CAE$, we have:

$AB = AC$ (Given)

$\angle BAE = \angle CAE$ (From equation 1)

$AE = AE$ (Common side)

So, by the SAS congruence rule, we have:

$\triangle BAE \cong \triangle CAE$

$\Rightarrow BE = CE$ and $\angle BEA = \angle CEA$ (By CPCT)

We know that $\angle BEA + \angle CEA = 180^\circ$ as they form a linear pair.

So, $2\angle BEA = 180^\circ$ (Proved above that $\angle BEA = \angle CEA$)

$\Rightarrow \angle BEA = 90^\circ$

Therefore, $\angle BEA = \angle CEA = 90^\circ$

Since $BE = CE$ and $\angle BEA = \angle CEA = 90^\circ$, AD is the perpendicular bisector of BC .

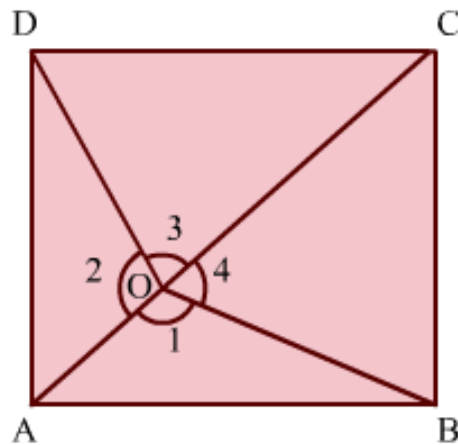
Example 2:

O is a point inside a square $ABCD$ such that it is at an equal distance from points B and D . Prove that points A , O and C are collinear.

Solution:

The square with the given specifications is drawn as is shown.

Construction: Join point O to the vertices of the square.



In $\triangle AOD$ and $\triangle AOB$, we have:

$AD = AB$ (Sides of a square)

$AO = AO$ (Common side)

$OD = OB$ (Given)

So, by the SSS congruence rule, we have:

$\triangle AOD \cong \triangle AOB$

$\Rightarrow \angle 1 = \angle 2 \dots (1)$ [By CPCT]

Similarly, $\triangle DOC \cong \triangle BOC$

$\Rightarrow \angle 3 = \angle 4 \dots (2)$ [By CPCT]

We know that:

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^\circ$$

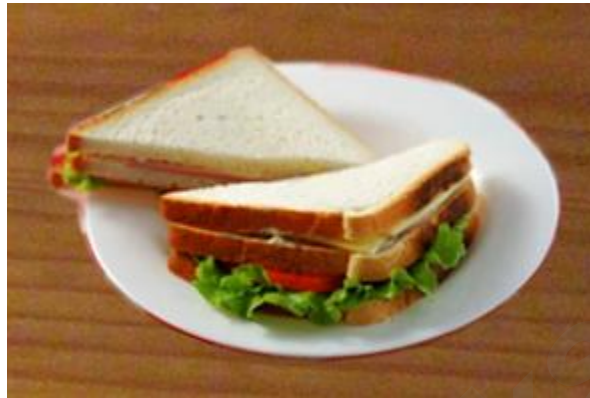
$$\Rightarrow 2\angle 2 + 2\angle 3 = 360^\circ \text{ (From equations 1 and 2)}$$

$$\Rightarrow \angle 2 + \angle 3 = 180^\circ$$

Thus, $\angle 2$ and $\angle 3$ form a linear pair. Therefore, AOC is a line; in other words, points A, O and C are collinear.

RHS Congruence Rule

Right angles are all around us. For example, while building houses, the walls are kept at a right angle to the horizontal. Different square and rectangular figures surround us and all of them consist of right angles. The following figure also shows right angles.

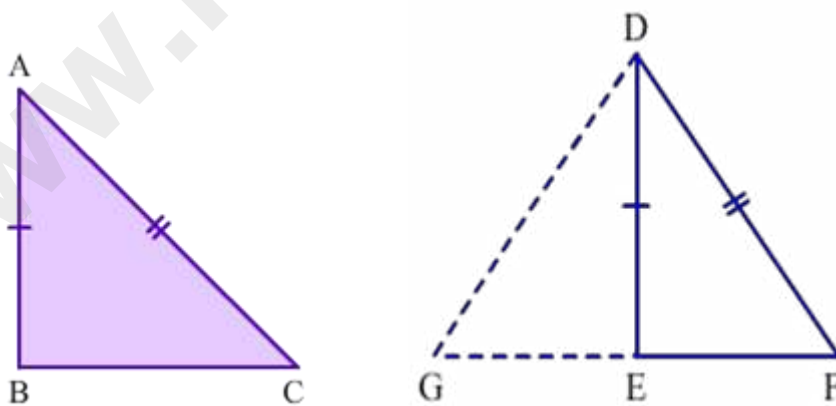


In the figure, the pieces of bread resemble right-angled triangles. They also seem to be congruent. Right-angled triangles are special and their congruency is checked by a special congruence rule known as the RHS (Right-Hypotenuse-Side) rule.

We will study the RHS congruence rule in this lesson and solve some examples to familiarize ourselves with the concept.

RHS Congruence Rule

RHS congruence theorem: Two right-angled triangles are congruent if the hypotenuse and a side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.



Given: Two right-angled triangle ABC and DEF such that $\angle B = \angle E = 90^\circ$; Hypotenuse $AC =$ Hypotenuse DF and $AB = DE$.

To prove: $\triangle ABC \cong \triangle DEF$.

Construction: Produce FE to G so that EG = BC and join DG.

Proof:

In triangles ABC and DEF,
AB = DE (Given)
BC = EG (By construction)
 $\angle ABC = \angle DEF$ (Each equal to 90°)

Thus, by SAS congruence criterion,
 $\triangle ABC \cong \triangle DEG$

$\Rightarrow \angle ACB = \angle DGE$ and $AC = DG$ (CPCT)
Given, $AC = DF$
 $\therefore DG = AC = DF$

In $\triangle DGF$, we have
 $DG = DF$
 $\angle G = \angle F$ (Angles opposite to equal sides are equal)

In $\triangle DEF$ and $\triangle DEG$,
 $\angle G = \angle F$ (Proved)
 $\angle DEG = \angle DEF$ (Both equal to 90°)
Thus, $\angle GDE = 180^\circ - (\angle G + \angle DEG) = 180^\circ - (\angle F + \angle DEF) = \angle FDE$

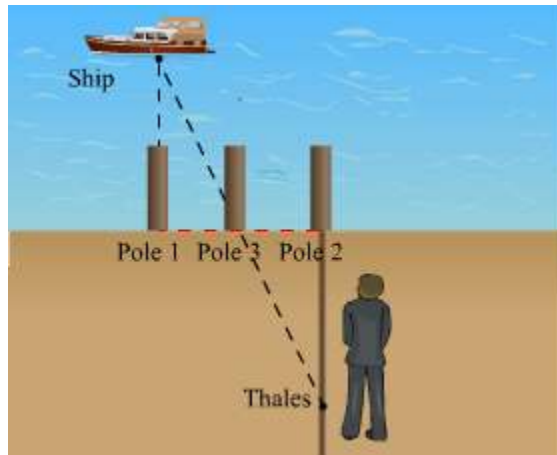
In $\triangle DEG$ and $\triangle DEF$,
 $DG = DF$ (Proved)
 $DE = DE$ (Common)
 $\angle GDE = \angle FDE$ (Proved)

Thus, by SAS congruence criterion
 $\triangle DEG \cong \triangle DEF$
But, we have $\triangle ABC \cong \triangle DEG$
Thus, $\triangle ABC \cong \triangle DEF$

Whiz Kid

RHS congruence rule was used by the famous ancient Greek mathematician Thales to calculate the distance of a ship anchored at sea from the shore. For this, he stuck three poles on the shore such that the first one was directly in front of the ship, the second was at some distance from the first pole and the third was exactly between the other two poles.

He then walked backward along a line from the second pole perpendicular to the shore until the middle pole and the ship were in the same line of sight. Then, he marked his position. This is shown in the following figure.



It can be seen that the triangle formed on the sea is congruent to the triangle formed on the shore by the RHS rule. So, the distance between the ship and the shore is equal to the distance between the second pole and the spot where Thales stands.

Solved Examples

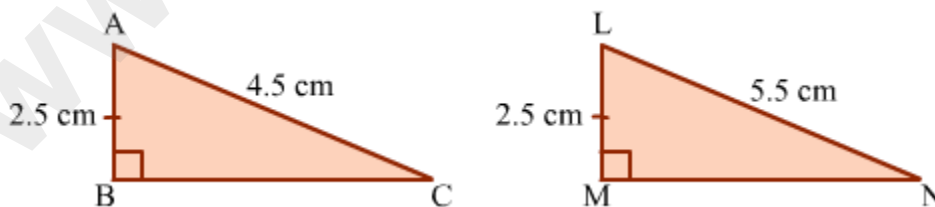
Easy

Example 1:

$\triangle ABC$ and $\triangle LMN$ are right-angled at $\angle ABC$ and $\angle LMN$ respectively. In $\triangle ABC$, $AB = 2.5$ cm and $AC = 4.5$ cm. In $\triangle LMN$, $LN = 5.5$ cm and $LM = 2.5$ cm. Examine whether the two triangles are congruent.

Solution:

On the basis of the given information, the two triangles can be drawn as is shown.



In $\triangle ABC$ and $\triangle LMN$, we have:

$$\angle ABC = \angle LMN \text{ (Right angles)}$$

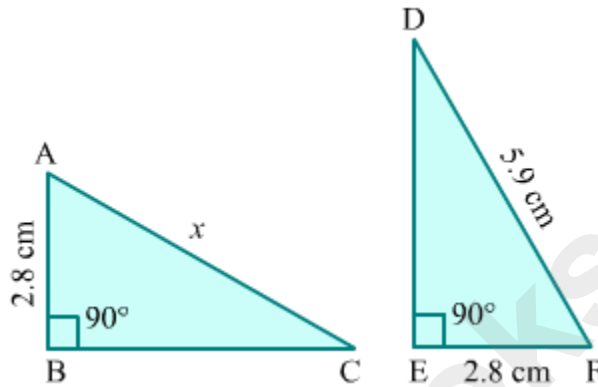
$$AB = LM = 2.5 \text{ cm (Given)}$$

But $AC \neq LN$

Hence, $\triangle ABC$ and $\triangle LMN$ are not congruent.

Example 2:

Find the value of x if the shown triangles ABC and DEF are congruent.



Solution:

It is given that $\triangle ABC \cong \triangle DEF$.

When two triangles are congruent, their corresponding sides are equal.

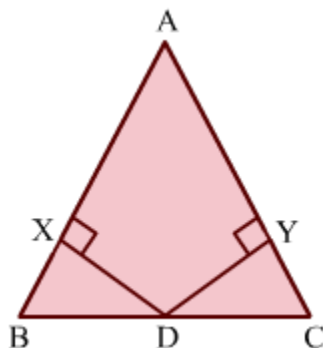
$$\therefore AC = DF = 5.9 \text{ cm}$$

Thus, the value of x is 5.9 cm.

Medium

Example 1:

In the given $\triangle ABC$, D is the midpoint of side BC . The perpendiculars DX and DY drawn from point D to sides AB and AC respectively are of the same length. Prove that DX and DY make the same angle with BC .



Solution:

On comparing $\triangle DXB$ and $\triangle DYC$, we get:

$$DX = DY \text{ (Given)}$$

$$\angle DXB = \angle DYC = 90^\circ \text{ } (\because DX \text{ and } DY \text{ are perpendiculars)}$$

$$BD = CD \text{ } (\because D \text{ is the midpoint of } BC)$$

Thus, by the RHS congruence rule, we have:

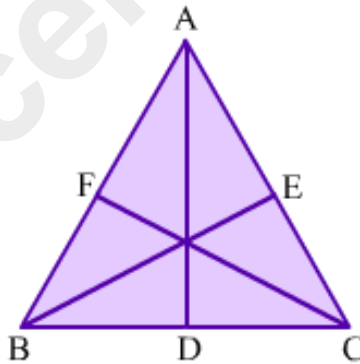
$$\triangle DXB \cong \triangle DYC$$

$$\Rightarrow \angle BDX = \angle CDY \text{ (By CPCT)}$$

Thus, the perpendiculars DX and DY make the same angle with side BC .

Example 2:

In the given $\triangle ABC$, AD , BE and CF are the altitudes. If the three altitudes are equal, then prove that the triangle is equilateral.



Solution:

In $\triangle BEC$ and $\triangle CFB$, we have:

$$BC = CB \text{ (Common side)}$$

$$BE = CF \text{ (Given)}$$

$$\angle BEC = \angle CFB = 90^\circ \text{ } (\because BE \text{ and } CF \text{ are altitudes)}$$

So, by the RHS congruence rule, we obtain:

$$\triangle BEC \cong \triangle CFB$$

$$\Rightarrow \angle BCE = \angle CBF \text{ (By CPCT)}$$

$$\Rightarrow \angle CBA = \angle BCA$$

$$\Rightarrow AC = AB \dots (1) \text{ [}\because \text{Sides opposite equal angles are equal]}$$

Similarly, we can prove that $\triangle ADB \cong \triangle BEA$.

$$\Rightarrow \angle DBA = \angle BAE \text{ (By CPCT)}$$

$$\Rightarrow \angle CBA = \angle BAC$$

$$\Rightarrow AC = BC \dots (2) \text{ [}\because \text{Sides opposite equal angles are equal]}$$

From equations (1) and (2), we get:

$$AB = BC = AC$$

Hence, $\triangle ABC$ is equilateral.

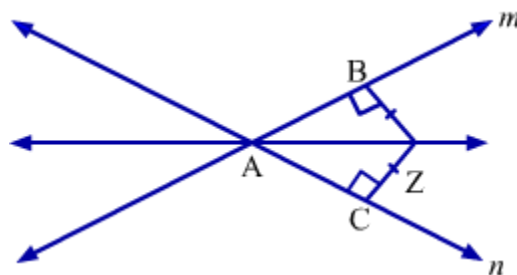
Hard

Example 1:

If Z is a point equidistant from two lines m and n intersecting at point A , then prove that AZ bisects the angle between m and n .

Solution:

The following figure can be drawn as per the given information.



Construction: ZB and ZC are perpendiculars drawn from point Z to lines m and n respectively.

Since Z is equidistant from m and n , we have:

$$ZB = ZC$$

In ΔZBA and ΔZCA , we have:

$$ZB = ZC \text{ (Proved above)}$$

$$\angle ZBA = \angle ZCA = 90^\circ \text{ } (\because ZB \text{ and } ZC \text{ are perpendiculars)}$$

$$ZA = ZA \text{ (Common side)}$$

So, by the RHS congruence rule, we have:

$$\Delta ZBA \cong \Delta ZCA$$

$$\Rightarrow \angle ZAB = \angle ZAC \text{ (By CPCT)}$$

$$\text{Now, } \angle ZAB + \angle ZAC = \angle BAC$$

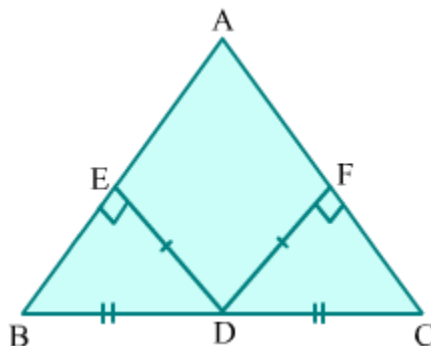
Therefore, AZ bisects the angle between lines m and n .

Example 2:

In a ΔABC , $BD = DC$. If the perpendiculars from point D to sides AB and AC are equal, then prove that $AB = AC$.

Solution:

The triangle with the given specifications is drawn below.



In ΔABC , D is the midpoint of BC . Also, DE and DF are the perpendiculars from D to AB and AC respectively.

In $\triangle DEB$ and $\triangle DFC$, we have:

$$\angle DEB = \angle DFC = 90^\circ \quad (\because DE \text{ and } DF \text{ are perpendiculars})$$

$$DB = DC \quad (\text{Given})$$

$$DE = DF \quad (\text{Given})$$

So, by the RHS congruence rule, we obtain:

$$\triangle DEB \cong \triangle DFC$$

$$\Rightarrow \angle DBE = \angle DCF \quad (\text{By CPCT})$$

$$\Rightarrow \angle CBA = \angle BCA$$

$$\Rightarrow AC = AB \quad (\text{As sides opposite to equal angles are equal})$$