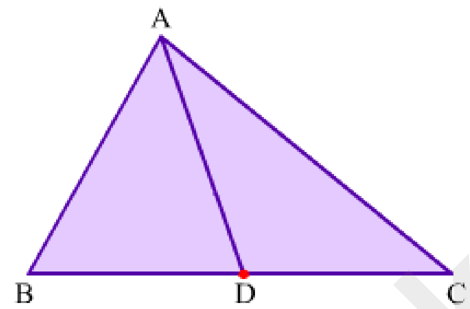
Triangles

Medians of Triangles

Let us consider the following triangle ABC.



In the given figure, A is the vertex of the triangle ABC and $\overline{BC}$ is the side opposite to vertex A. A line segment $\overline{AD}$ is drawn joining the point A and the point D, where D is the mid-point of $\overline{BC}$.

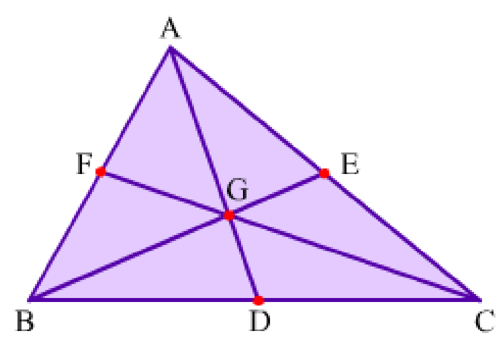
Then, we say that $\overline{AD}$ **is the median of ΔABC.**

A median can be defined as follows.

"The line segment joining any vertex of a triangle to the mid-point of its opposite side is called the median of the triangle."

Now we know what a median is, can we tell how many medians can be drawn inside a triangle?

In a triangle, there are three vertices. Therefore, ***a triangle can have three medians***, as shown in the following figure.



Here, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the three medians of ΔABC.

The medians of a triangle always lie inside the triangle.

From the figure, it can be observed that the medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect each other at a common point G.

"The point of intersection of the medians is called the **centroid** of the triangle."

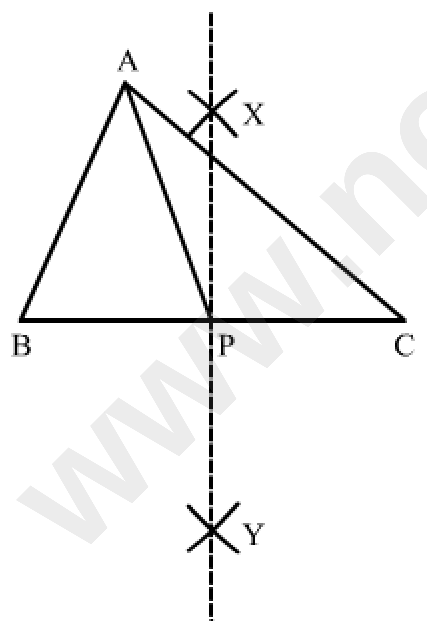
Thus, medians of a triangle are concurrent.

The point where medians intersect each other is known as the point of concurrence.

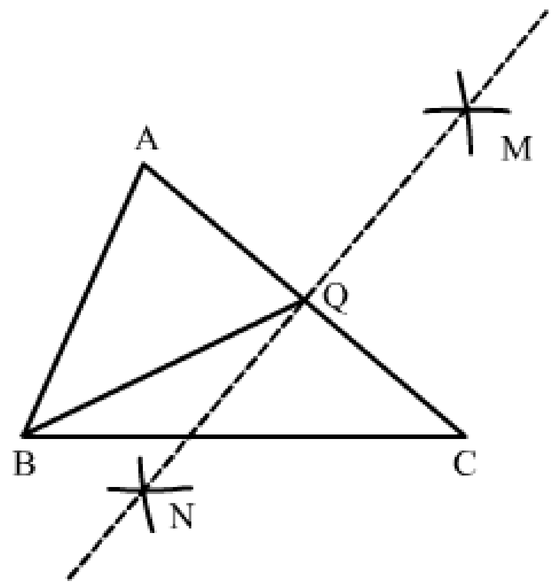
In the above given figure, G is the point of concurrence.

Construction of Median of Triangle

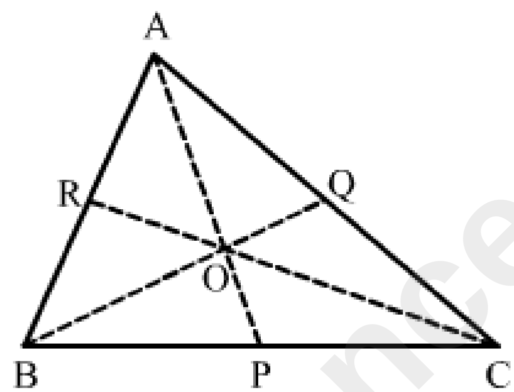
1. Draw a △ABC.
2. With B and C as centres and radius more than half of BC, draw two arcs intersecting at points X and Y. Join XY thus meeting the line BC at point P.



3. With A and C as centres and radius more than half of AC, draw two arcs intersecting at points M and N. Join MN thus meeting the line AC at point Q.



4. Similarly, draw the perpendicular bisector of line AB meeting AB at point R.
5. Join AP, BQ and CR. Let the meeting point be O.

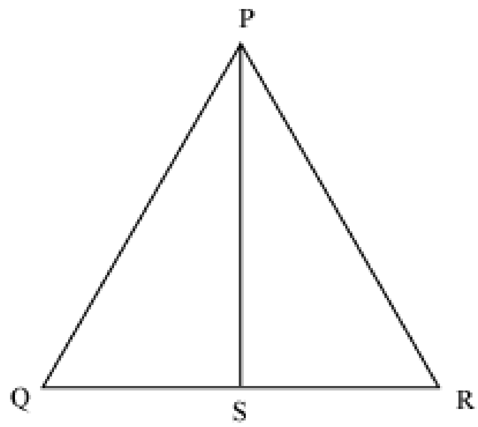


Point O is the centroid of △△ABC and AP, BQ and CR are the medians of sides BC, AC and AB respectively.

Now, let us look at an example.

Example 1:

In the triangle PQR, PS is a median and the length of $\overline{SR}$ = 6.5 cm. Find the length of $\overline{QR}$.



Solution:

Here, PS is the median to the side $\overline{QR}$ and we know that the median connects vertex to the midpoint of other side. Therefore, S is the mid-point of QR.

Therefore, $\overline{QR} = 2\overline{SR}$

$= 2 \times 6.5$ cm

$= 13$ cm

Altitudes of Triangles

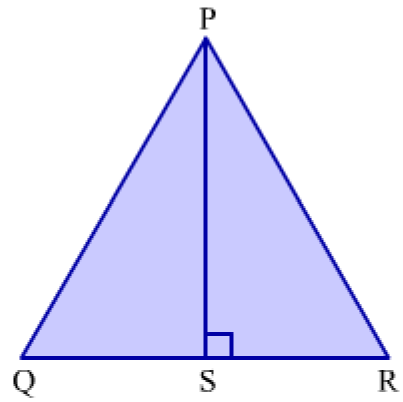
The students of class VII were being taken to a tour to Corbett National Park. They stayed there in a tent. The entrance in the tent was of a triangular shape as shown in the following figure.

Now can you tell what the height of the tent is?

As we can see in the above figure, the height of the tent is the length of the vertical pole which is standing in the centre of the tent.

Similarly, in any triangle, we can draw a perpendicular which represents its height. The perpendicular representing the **height** of a triangle is called the **altitude** of the triangle.

Look at the triangle PQR below.

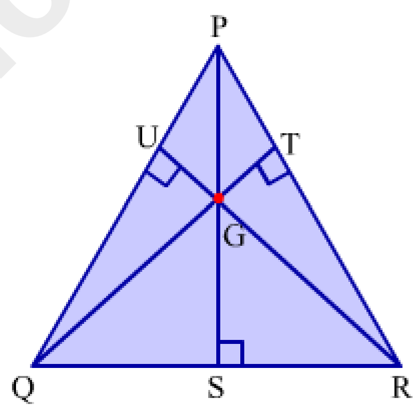


Here, P is a vertex of ΔPQR and $\overline{QR}$ is the opposite side of the vertex P. $\overline{PS}$ is a perpendicular drawn from P to $\overline{QR}$. Line segment $\overline{PS}$ is called the height or altitude of the triangle.

An altitude can be defined as follows.

"An altitude of a triangle is the perpendicular drawn from a vertex to the opposite side of the triangle."

Note: ***A triangle can have three altitudes.***



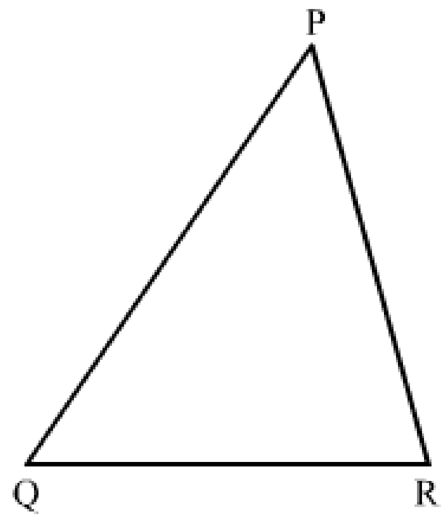
In the above figure, $\overline{PS}$, $\overline{QT}$, and $\overline{RU}$ are the three altitudes of ΔPQR.

"The point of intersection of the altitudes is called the **orthocentre** of the triangle."

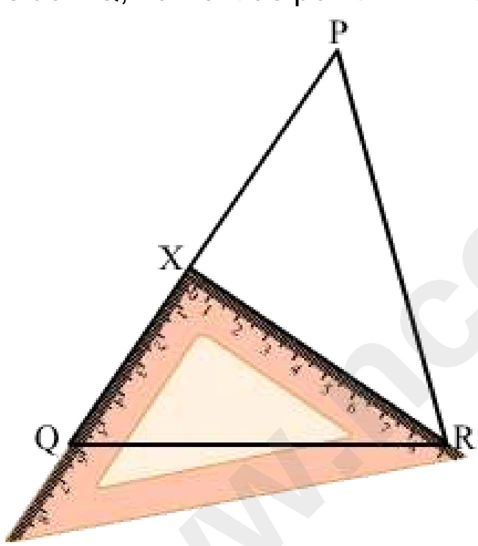
<u>Construction of Altitudes of a triangle</u>

I. Using set-square

1. Draw a △PQR.

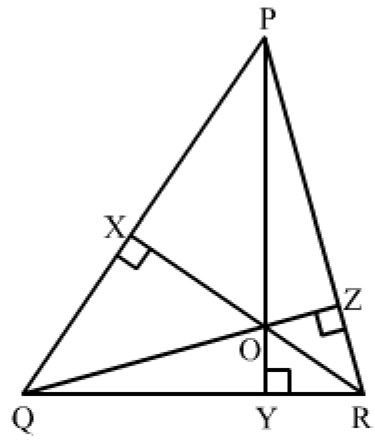


2. From point R, draw a perpendicular on side PQ. Where the perpendicular meets the side PQ, name it as point X. XR is the altitude formed on side PQ.



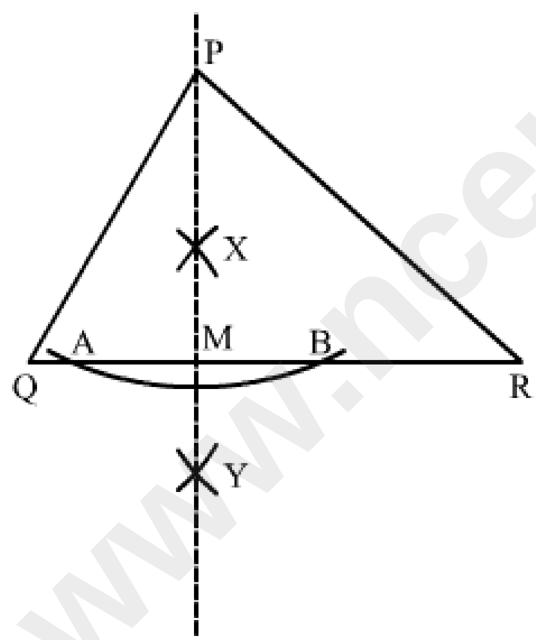
3. In a similar way construct the altitude from point P to side QR and from side Q to line PR.

Thus, the altitudes obtained are XR, QZ and PY.

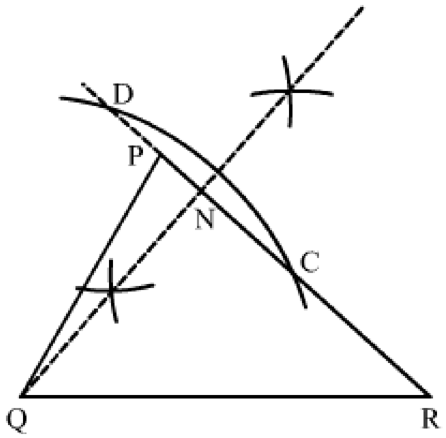


II. Using Compasses

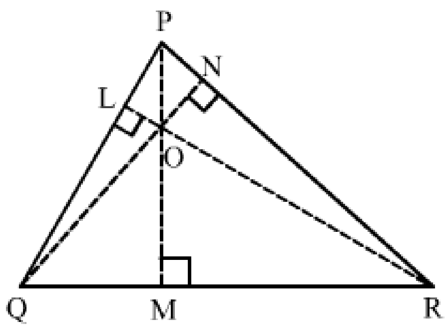
1. Draw a △△PQR.
2. With P as centre, draw an arc on line QR cutting it at points A and B.
3. With A and B as centres, draw two intersecting arcs at points X and Y. Draw a line joining XY cutting the line QR at point M. Join PM.



4. With Q as centre draw an arc on side RP extended to cut it at points C and D. With C and D as centres, draw two intersecting arcs. Let this line intersect PR at point N. Join QN.



5. Similarly, draw altitude from point R on PQ cutting the line PQ at point L.
6. Join PM, RL and QN and name the meeting point of these three altitudes as O.



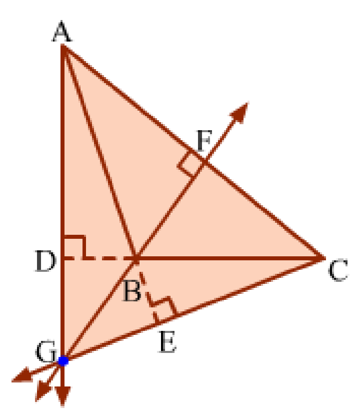
O is called the orthocentre of the △△PQR.

Remember

The altitudes of a triangle may not always lie inside it.

In an obtuse-angled triangle, the altitude drawn from the vertex of an acute angle lies outside the triangle. In this case, we have to extend the opposite side of the vertex from which the altitude is drawn.

For example,



In the above figure, ΔABC is an obtuse-angled triangle where ∠ABC is an obtuse angle. $\overline{AD}$ is the altitude of ΔABC drawn from the vertex A to extended side $\overline{BC}$. Similarly, $\overline{CE}$ is the altitude drawn from the vertex C to extended side $\overline{AB}$. And, $\overline{BF}$ is the altitude drawn from B to $\overline{CA}$.

Now, observe the altitudes drawn in the triangles ΔPQR and ΔABC.

It can be seen that altitudes in each triangle intersect each other at a common point.

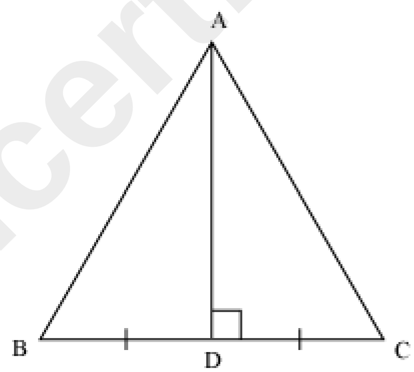
Thus, altitudes of a triangle are concurrent.

In ΔPQR and ΔABC, G is the point of concurrence.

Let us look at another example now.

Example:

In triangle ABC, $\overline{AD}$ is perpendicular to $\overline{BC}$ such that $\overline{BD} = \overline{CD}$. Are the median and the altitude drawn from A to $\overline{BC}$ same?



Solution:

Here,

$\overline{AD} \perp \overline{BC}$

Therefore, $\overline{AD}$ is an altitude of ΔABC drawn from the vertex A to $\overline{BC}$.

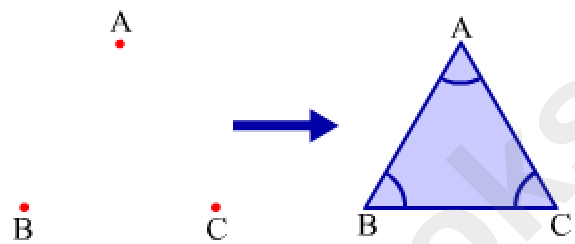
Also, $\overline{BD} = \overline{CD}$

Therefore, $\overline{AD}$ is a median of ΔABC drawn from the vertex A to $\overline{BC}$.

Thus, the altitude and the median drawn from A to $\overline{BC}$ are the same.

Angle Sum Property of Triangles

If we join any three non-collinear points in a plane, then we **get a triangle. There are three angles in a triangle.**



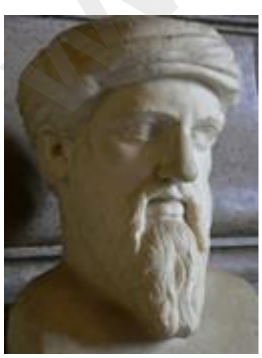
The sum of the three **interior angles** of a triangle is 180° and this property of a triangle is known as the angle sum property. This property holds true for all types of triangles, i.e., **acute-angled triangles**, **obtuse-angled triangles** and **right-angled triangles**. The angle sum property was identified by the Pythagorean school of Greek mathematicians (or the Pythagoreans) and proved by Euclid.

We will study the proof of the angle sum property of triangles and then solve some examples based on this property.

Proving the Angle Sum Property of Triangles

Know Your Scientist

Pythagoras



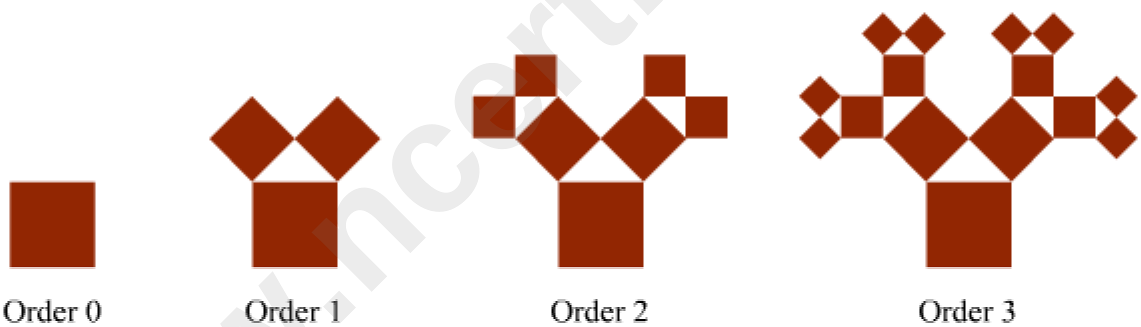
Pythagoras (570 BC−495 BC) was a great Greek mathematician and philosopher, often described as the first pure mathematician. He was born on the island of Samos and is best known for the Pythagoras theorem about right-angled triangles. He also made influential contributions to philosophy and religious teaching. He led a society that was part religious and part scientific. This society followed a code of secrecy, which is the reason why a sense of mystery surrounds the figure of Pythagoras.

Euclid

Euclid of Alexandria(325 BC−265 BC) was a great Greek mathematician. He is referred to as 'the father of geometry'. Euclid taught at Alexandria during the reign of Ptolemy I, who ruled Egypt from 323 BC to 285 BC. Euclid wrote a series of books which are collectively known as the *Elements*.It is considered one of the most influential works in the history of mathematics.
The *Elements* served as the main textbook for teaching mathematics (especially geometry) from the time of its publication up until the early 20th century. In the *Elements*, Euclid defined most of the basic geometrical figures and deduced the principles of geometry through different sets of axioms.

Did You Know?

In 1942, a Dutch mathematics teacher Albert E. Bosman invented a plane fractal constructed from a square. He named it the Pythagoras tree because of the presence of right-angled triangles in the figure.



Construction process of Pythagoras tree

Facts about the Angle Sum Property

An important fact deduced through the angle sum property of triangles is that ***there can be no triangle with two right angles or two obtuse angles*. This fact can be proved as is shown.**

Consider a ΔABC such that ∠A = 90° and ∠B = 90°.

According to the angle sum property, we have:

∠A + ∠B + ∠C = 180°

⇒ 90° + 90° + ∠C = 180°

⇒ ∠C = 180° − 180°

⇒ ∠C = 0°

However, the above is not possible. So, ΔABC (or any other triangle) cannot have two right angles.

Similarly, we can prove that a triangle cannot have two obtuse angles.

Whiz Kid

Relationship between the side lengths and the angle measurements of a triangle

- The largest interior angle is **opposite** the largest side.
- The smallest interior angle is **opposite** the smallest side.
- The middle-sized interior angle is **opposite** the middle-sized side.

Facts about the Angle Sum Property

By the angle sum property, we can deduce the fact that ***there can be no triangle with all angles less than or greater than*** **60°.** This fact can be proved as is shown.

Consider a **ΔABC** with all angles equal to 59°.

According to the angle sum property, we should have **∠A + ∠B + ∠C = 180°.**

By adding the given angles, we obtain:

59° + 59° + 59° **= 177° ≠ 180°**

Since ΔABC does not satisfy the angle sum property, it cannot exist.

Now, consider a **ΔABC** with all angles equal to 61°.

According to the angle sum property, we should have **∠A + ∠B + ∠C = 180°.**

By adding the given angles, we obtain:

61° **+** 61° **+** 61° **= 183° ≠ 180°**

Since ΔABC does not satisfy the angle sum property, it cannot exist.

Thus, we have proved that a triangle cannot have all angles less than or greater than 60°.

Whiz Kid

Sum of the interior angles of an n-sided polygon = $(n - 2) \times 180°$

For example:

Sum of the interior angles of a 6-sided polygon = $(6 - 2) \times 180° = 720°$

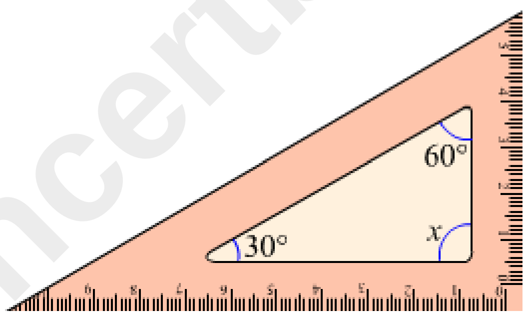
Relation between the Vertex Angle and the Angles Made by the Bisectors of the Remaining Angles

Solved Examples

Easy

Example 1:

Find the measurement of *x* in the following figure.



Solution:

We know that the sum of the three angles of a triangle is 180°.

So, we have:

$30° + 60° + x = 180°$

$\Rightarrow 90° + x = 180°$

$\Rightarrow x = 90°$

Example 2:

If the angles of a triangle are in the ratio 1 : 3 : 5, then what is the measure of each angle?

Solution:

It is given that the angles of the triangle are in the ratio 1 : 3 : 5.

Let the angles be x, $3x$ and $5x$.

Now, $x + 3x + 5x = 180°$ (By the angle sum property of triangles)

$\Rightarrow 9x = 180°$

$\Rightarrow x = 20°$

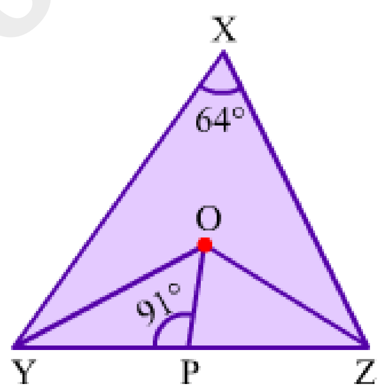
So, $3x = 3 \times 20° = 60°$ and $5x = 5 \times 20° = 100°$

Thus, the measures of the angles of the triangle are 20°, 60° and 100°.

Medium

Example 1:

In the given ΔXYZ, YO, ZO and PO are the respective bisectors of ∠XYZ, ∠XZY and ∠YOZ. Find the measure of ∠OYX.



Solution:

As per the relation between the vertex angle and the angles made by the bisectors of the remaining two angles, we have:

$$\angle YOZ = 90° + \frac{1}{2}\angle YXZ$$
$$\Rightarrow \angle YOZ = 90° + \frac{1}{2} \times 64°$$
$$\Rightarrow \angle YOZ = 90° + 32°$$
$$\Rightarrow \angle YOZ = 122°$$

In ΔOYP, we have:

∠YOP = $\frac{1}{2}$ ∠YOZ (&because PO is the bisector of ∠YOZ)

⇒ ∠YOP = $\frac{1}{2}$ × 122°

⇒ ∠YOP = 61°

Again in ΔOYP, we have:

∠YOP + ∠OPY + ∠PYO = 180° (By the angle sum property)

⇒ 61° + 91° + ∠PYO = 180°

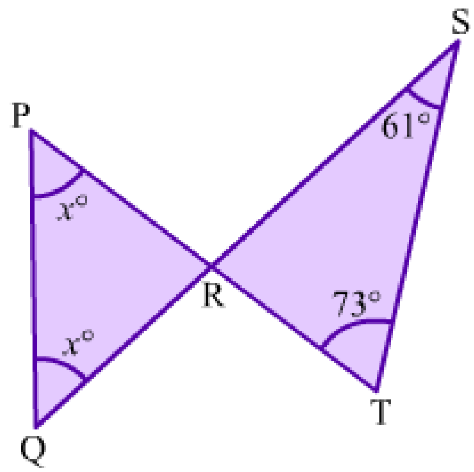
⇒ ∠PYO = 180° − 152°

⇒ ∠PYO = 28°

We know that YO is the bisector of ∠XYZ.

So, ∠OYX = ∠PYO = 28°

Example 2:
Find the value of *x* in the given figure.



Solution:

Using the angle sum property in ΔRST, we obtain:

∠RST + ∠RTS + ∠SRT = 180°

⇒ 61° + 73° + ∠SRT = 180°

⇒ 134° + ∠SRT = 180°

⇒ ∠SRT = 180° − 134°

⇒ ∠SRT = 46°

In ΔRST and ΔRPQ, we have:

∠PRQ = ∠SRT = 46° (Vertically opposite angles)

Using the angle sum property in ΔRPQ, we obtain:

∠RPQ + ∠RQP + ∠PRQ = 180°

⇒ x° + x° + 46° = 180°

⇒ 2x° + 46° = 180°

⇒ 2x° = 180° − 46°

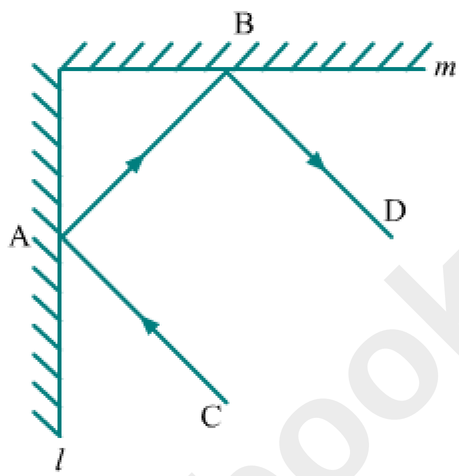
⇒ 2x° = 134°

⇒ x = 67

Hard

Example 1:

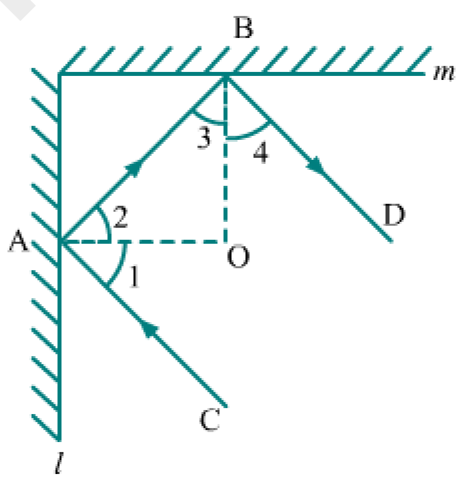
In the figure, *l* and *m* are two plane mirrors placed perpendicular to each other. Show that the incident ray CA is parallel to the reflected ray BD.



Solution:

It is given that mirrors *l* and *m* are perpendicular to each other.

Construction: Draw two perpendiculars OA and OB to *l* and *m* respectively. Mark the angles made by these perpendiculars with the incident and reflected rays as is shown.



OA ⊥ OB

∴ **∠BOA = 90°**

In ΔBOA, we have:

∠2 + ∠3 + ∠BOA = 180°

⇒ ∠2 + ∠3 + 90° = 180°

⇒ ∠2 + ∠3 = 90°

⇒ 2(∠2 + ∠3) = 180°

⇒ 2∠2 + 2∠3 = 180° **… (1)**

We know that:

Angle of incidence = Angle of reflection

For mirror *l*, ∠1 is the angle of incidence and ∠2 is the angle of reflection.

∴ ∠1 = ∠2

For mirror *m*, ∠3 is the angle of incidence and ∠4 is the angle of reflection.

∴ ∠3 = ∠4

So, by using equation 1, we get:

(∠2 + ∠2) + (∠3 + ∠3) = 180°

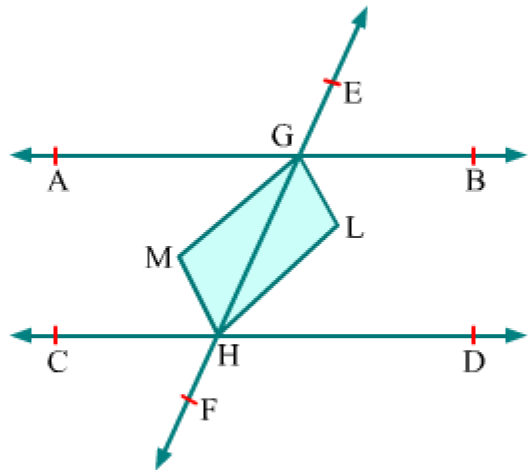
⇒ (∠1 + ∠2) + (∠3 + ∠4) = 180° **(**Since ∠1 = ∠2 and ∠3 = ∠4**)**

⇒ ∠CAB + ∠ABD = 180°

Since ∠CAB and ∠ABD are interior angles on the same side of transversal AB and their sum is 180°**, lines CA and BD must be parallel to each other. Therefore, the incident ray CA is parallel to the reflected ray BD.**

Example 2:

In the given figure, AB is parallel to CD; GM, HM, GL and HL are the bisectors of the two pairs of interior angles. Prove that ∠GLH = 90°.



Solution:

From the figure, we have:

∠AGH = ∠DHG (Alternate interior angles)

⇒ $\frac{1}{2}$ ∠AGH = $\frac{1}{2}$ ∠DHG

⇒ ∠HGM = ∠GHL (&because GM bisects ∠AGH and HL bisects ∠DHG)

It can be said that lines GM and HL are intersected by the transversal GH at G and H respectively such that the alternate interior angles are equal, i.e., ∠HGM = ∠GHL.

∴ GM||HL

Similarly, we can prove that GL||HM. So, GMHL is a parallelogram.

We know that AB||CD and EF is the transversal.

∴ ∠BGH + ∠DHG = 180° (Interior angles on the same side of a transversal)

⇒ $\frac{1}{2}$ ∠BGH + $\frac{1}{2}$ ∠DHG = 90°

Since GL bisects ∠BGH and HL bisects ∠DHG, we obtain:

∠LGH + ∠GHL = 90° … (1)

Also, ∠LGH + ∠GHL + ∠GLH = 180° (By the angle sum property)

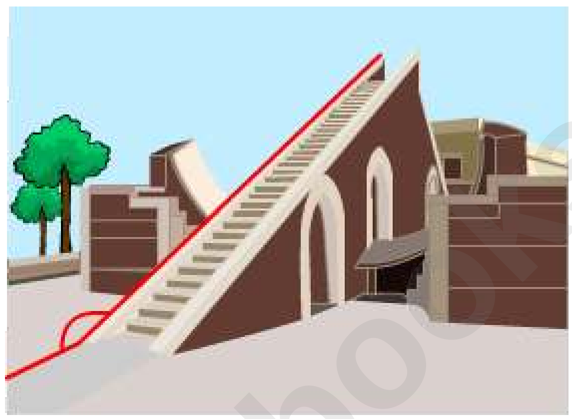
∴ 90° + ∠GLH = 180° (Using equation 1)

⇒ ∠GLH = 90°

Exterior Angle Property of Triangles

Exterior Angles in Real Life

Look at the triangular structure in the figure.

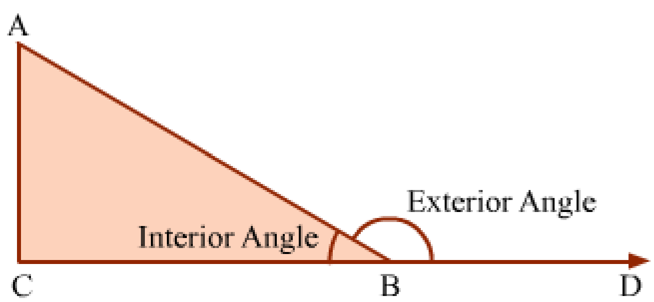


In the given figure, an open angle formed by the edge of the triangular structure with the horizontal plane is marked. This angle lies outside the triangle. Such angles are known as **exterior angles**.

In this lesson, we will study about exterior angles of triangles and the theorem based on them.

Exterior Angles of Triangles

Look at the triangle shown.



It can be seen that in **Δ**ABC, side CB is extended up to point D. This extended side forms an angle with side AB, i.e., ∠ABD. This angle lies exterior to the triangle. Hence, ∠**ABD is an exterior angle of ΔABC.**

An exterior angle of a triangle can be defined as follows:

The angle formed by a side of a triangle with an extended adjacent side is called an exterior angle of the triangle.

It can be seen that exterior ∠ABD forms linear pair with interior ∠ABC of **Δ**ABC. The other two interior angles of the triangle such as ∠ACB and ∠CAB do not form linear pair with ∠ABD.

Such angles are known as the **remote interior angles** of an exterior angle.

So, ∠ACB and ∠CAB are remote interior angles of exterior ∠ABD.

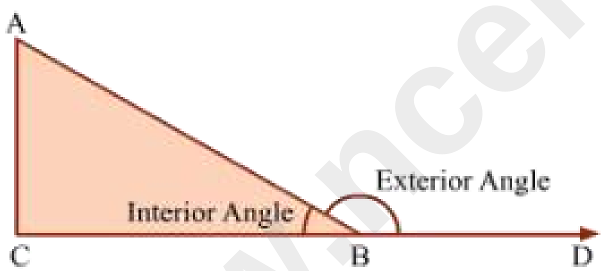
Exterior Angle Theorem or Remote Interior Angles Theorem and Its Proof

Corollary Related to Exterior Angle Theorem

There is a corollary related to exterior angle theorem which states that:

An exterior angle is always greater than each of its remote interior angles.

Let us prove this corollary with the help of **Δ**ABC shown in the figure.



Here, ∠ABD is an exterior angle of the triangle and its interior opposite or remote interior angles are ∠ACB and ∠CAB.

In a triangle, no interior angle can be zero angle or straight angle.

Thus, 0° <∠ABC < 180°, 0° < ∠ACB < 180° and 0° < ∠CAB < 180°

Now, ∠ABC < 180°

∴ 180° – ∠ABC > 0°

⇒ ∠ABD > 0°

In triangle **Δ**ABC, we have

∠ABD = ∠ACB +∠CAB (By exterior angle theorem)

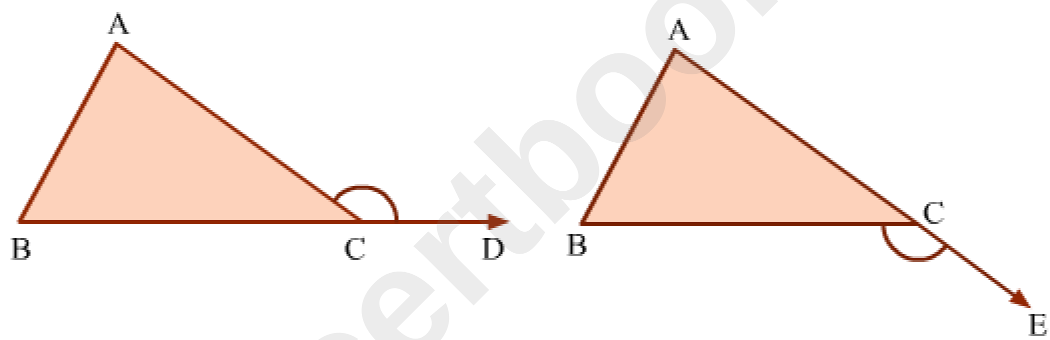
And, ∠ABD > 0, ∠ACB > 0° and ∠CAB > 0° (Property of triangle)

Therefore, ∠ABD > ∠ACB and ∠ABD >∠CAB

Thus, an exterior angle is always greater than each of its remote interior angles.

Two Exterior Angles at the Same Vertex are Equal

At any vertex, two exterior angles can be drawn by extending each of the two sides forming that vertex. These exterior angles are always of equal measure. Let us prove this using the ΔABC shown in the figure.



The figure clearly shows that two exterior angles can be drawn at vertex C—one by producing BC up to point D and the other by producing AC up to point E. The exterior angles thus obtained are ∠ACD and ∠BCE.

According to the exterior angle theorem, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

∴ ∠ACD = ∠ABC + ∠BAC … (1)

And, ∠BCE = ∠ABC + ∠BAC … (2)

Using equations 1 and 2, we get:

∠ACD = ∠BCE

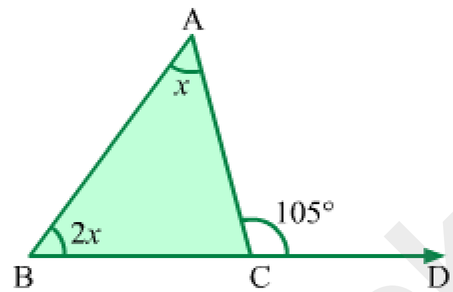
So, we can conclude that two exterior angles can be drawn at any vertex. The two angles thus drawn have an equal measure and are equal to the sum of the two opposite interior angles.

Solved Examples

Easy

Example 1:

Find the value of *x* in the given figure.



Solution:

According to the exterior angle property of triangles, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

So, we have:

$x + 2x = 105°$

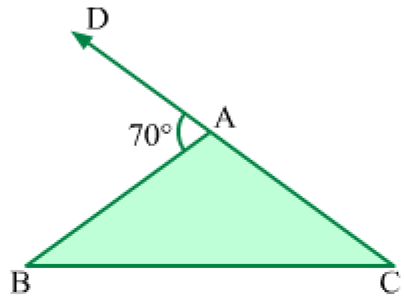
$\Rightarrow 3x = 105°$

On dividing both sides of the equation by 3, we obtain:

$$\frac{3x}{3} = \frac{105°}{3}$$
$$\Rightarrow x = 35°$$

Example 2:

If ∠ABC = ∠ACB in ΔABC, then find the measure of ∠ABC.



Solution:

∠ABC and ∠ACB are interior angles opposite to the exterior angle at vertex A, i.e., ∠BAD.

Therefore, by the exterior angle property of triangles, we obtain:

∠ABC + ∠ACB = ∠BAD

⇒ ∠ABC + ∠ACB = 70°

It is given that ∠ABC = ∠ACB

So, we obtain:

∠ABC + ∠ABC = 70°

⇒ 2∠ABC = 70°

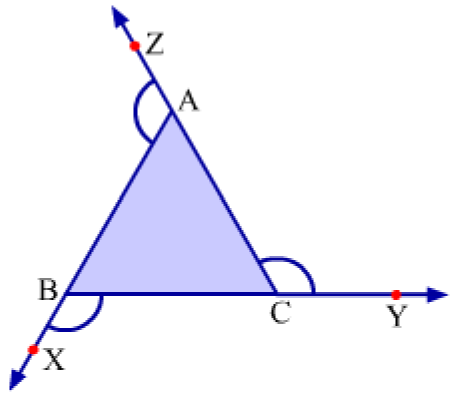
On dividing both sides of the equation by 2, we obtain:

$$\frac{2\angle ABC}{2} = \frac{70^\circ}{2}$$
$$\Rightarrow \angle ABC = 35^\circ$$

Medium

Example 1:

The sides AB, BC and CA of ΔABC are produced up to points X, Y and Z respectively. Find the sum of the three exterior angles so formed.



Solution:

Using the exterior angle property, we obtain:

∠BAZ = ∠ABC + ∠ACB … (1)

∠CBX = ∠BAC + ∠ACB … (2)

∠ACY = ∠BAC + ∠ABC … (3)

On adding equations 1, 2 and 3, we obtain:

∠BAZ + ∠CBX + ∠ACY = ∠ABC + ∠ACB + ∠BAC + ∠ACB + ∠BAC + ∠ABC

⇒ ∠BAZ + ∠CBX + ∠ACY = 2(∠ABC + ∠ACB + ∠BAC)

According to the angle sum property of triangles, we have:

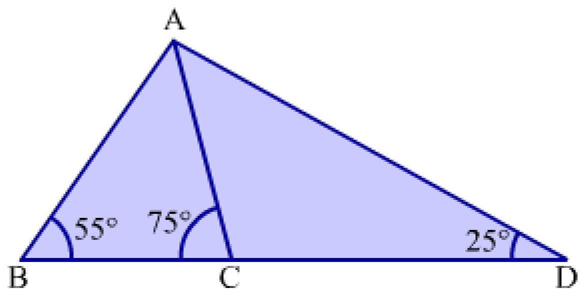
∠ABC + ∠ACB + ∠BAC = 180°

∴ ∠BAZ + ∠CBX + ∠ACY = 2 × 180° = 360°

Thus, the sum of the three exterior angles is 360°.

Example 2:

Show that AC is the bisector of ∠BAD in the given figure.



Solution:

On applying the angle sum property in ΔABC, we get:

∠BAC + ∠ABC + ∠ACB = 180°

⇒ ∠BAC + 55° + 75° = 180°

⇒ ∠BAC = 180° − 130°

⇒ ∠BAC = 50° … (1)

Now, by using the exterior angle property, we get:

∠ACB = ∠ADC + ∠CAD

⇒ 75° = 25° + ∠CAD

⇒ ∠CAD = 75° − 25°

⇒ ∠CAD = 50°

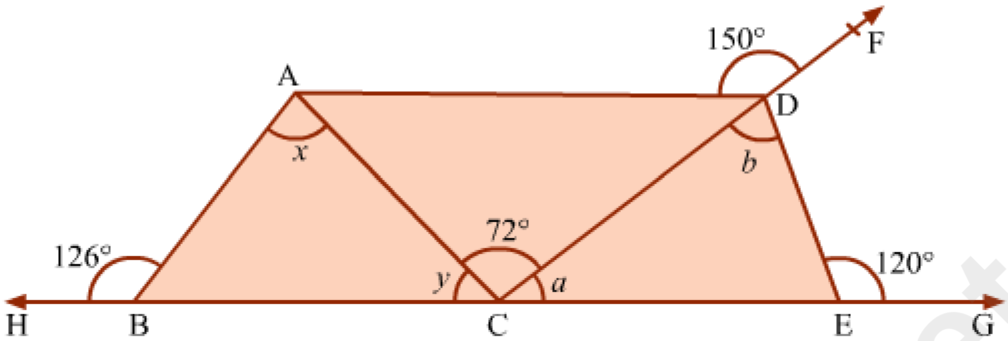
We know that ∠BAC + ∠CAD = ∠BAD. We have found ∠BAC = ∠CAD = 50°.

Thus, AC is the bisector of ∠BAD.

Hard

Example 1:

If AD||BE in the given figure, then find the values of *a*, *b*, *x* and *y*.



Solution:

From the figure, we have:

∠ADC + ∠ADF = 180° (Linear pair of angles)

⇒ ∠ADC + 150° = 180°

⇒ ∠ADC = 30°

Consider the parallel lines AD and BE and the transversal CF.

∠ADF = ∠DCB (Corresponding angles)

⇒ 150° = 72° + y

⇒ y = 78° … (1)

Now, y + 72° + a = 180° (As they form line BCE)

⇒ 78° + 72° + a = 180° (Using equation 1)

⇒ a = 180° − 150°

⇒ a = 30° … (2)

Consider ΔCDE.

∠DEG = a + b (Exterior angle property)

⇒ 120° = 30° + b (Using equation 2)

⇒ b = 90°

Now, consider ΔABC.

∠ABH = $x + y$ (Exterior angle property)

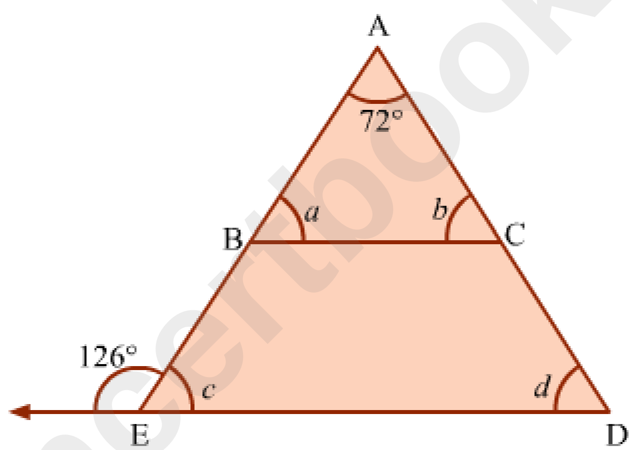
⇒ 126° = x + 78°

⇒ x = 48°

Hence, a = 30°, b = 90°, x = 48° and y = 78°.

Example 2:

ΔABC is placed atop trapezium EBCD in the given figure. Find the values of *a*, *b*, *c* and *d*.



Solution:

The exterior angle at E forms a linear pair with c.

∴ 126° + c = 180°

⇒ c = 180° − 126°

⇒ c = 54°

On using the exterior angle property in ΔAED, we get:

126° = 72° + d

⇒ d = 126° − 72°

⇒ d = 54°

Since EBCD is a trapezium, BC is parallel to ED. Also, BE and CD are transversals lying on the two parallel lines.

So, we have:

$a = c = 54°$ (Pair of corresponding angles)

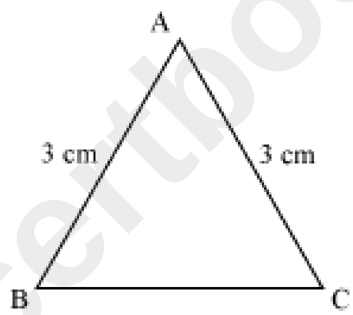
$b = d = 54°$ (Pair of corresponding angles)

Thus, $a = b = c = d = 54°$.

Isosceles Triangles

We can classify triangles into different categories according to its sides (or angles). Now, let us study about a special type of triangle known as isosceles triangle.

“A triangle in which two sides are of equal length is called isosceles triangle.”



In ΔABC, two sides AB and AC are of equal length. Therefore, ΔABC is an isosceles triangle.

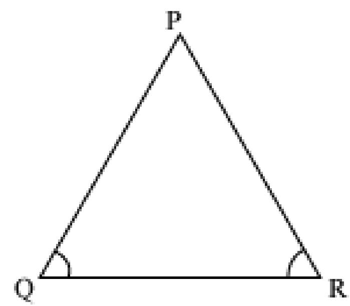
“In an isosceles triangle, angles opposite to equal sides are also equal.”

In the above figure, ∠B and ∠C are the angles opposite to equal sides AC and AB respectively.

$\therefore \angle B = \angle C$

Therefore, we can also define isosceles triangle in terms of angles.

“If two angles of a triangle are equal in measure, then the triangle is called isosceles triangle.”



In the given figure,

∠Q = ∠R

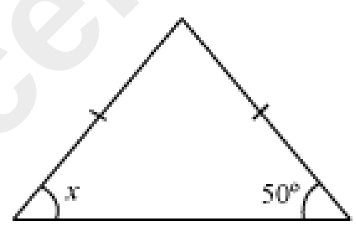
∴ ΔPQR is an isosceles triangle.

Now, let us solve some examples involving isosceles triangles.

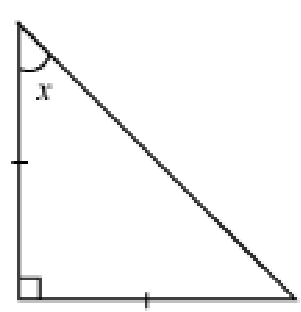
Example 1:

Find the value of *x* for the following figures.

(i)



(ii)



Solution:

(i) The given triangle is an isosceles triangle.

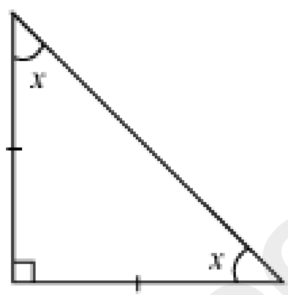
The angles opposite to equal sides are equal.

∴ $x = 50°$

(ii) The given triangle is an isosceles triangle.

The angles opposite to equal sides are equal.

Thus, we have



Now, using angle sum property of triangles, we obtain

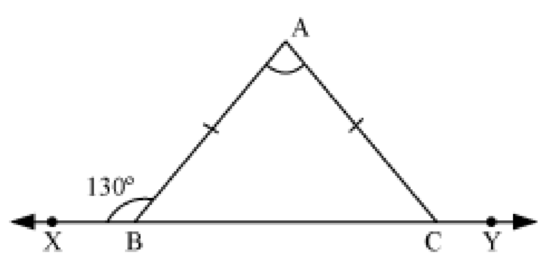
$x + x + 90° = 180°$

$2x = 90°$

$x = 45°$

Example 2:

In the given figure, find ∠BAC.



Solution:

It is given that ΔABC is an isosceles triangle. We know that in an isosceles triangle, angles opposite to equal sides are equal.

∴ ∠ABC = ∠ACB (1)

Now, $\angle ABC = 180° - \angle ABX$ (By linear pair axiom)

$= 180° - 130°$

$= 50°$

From equation (1), we obtain

$\angle ACB = 50°$

Using angle sum property in ΔABC, we obtain

$\angle ABC + \angle ACB + \angle BAC = 180°$

$50° + 50° + \angle BAC = 180°$

$\angle BAC = 180° - 100°$

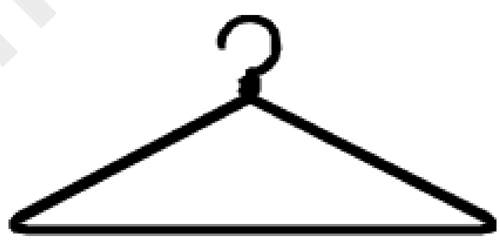
$\angle BAC = 80°$

Thus, the measure of $\angle BAC$ is 80°.

Angles Opposite to Equal Sides of an Isosceles Triangle are Equal

Isosceles Triangles around Us

You must have seen triangular clothes hangers or coat hangers such as the one shown.



What do you observe about the sides of the hanger? The base is clearly the longest side, while the other two sides are equal. We know that a triangle with any two sides equal is called an isosceles triangle. So, the hanger is a real-life example of an isosceles triangle. Another example of an isosceles triangle is the roof of a hut. Yet another example is a diagonally cut slice of bread.

You can see that the angles opposite the equal sides of the clothes hanger are equal. In this lesson, we will discuss about the same, i.e., angles opposite equal sides of an isosceles triangle. We will also solve some problems based on this concept.

Angles Opposite Equal Sides of an Isosceles Triangle Are Equal

Know Your Scientist

Thales (624 BC−546 BC) is believed to be the first philosopher in the Greek tradition. A great mathematician and scientist, he is regarded as 'the father of science'. Thales is credited for giving the first known mathematical proof, i.e., 'a circle is bisected by its diameter'. He was also the first to prove the theorem: 'any angle inscribed in a semicircle is a right angle'. For this reason, this theorem is called Thales Theorem.

Thales also generalized the concept of equality of the base angles of an isosceles triangle. He also propounded a few general notions such as 'all straight angles are equal', 'equals subtracted from equals are equal' and 'equals added to equals are equal'. These theorems have helped in the evolution of modern mathematics.

Did You Know?

The word 'isosceles' is a combination of the Greek words 'isos' meaning 'equal' and 'skelos' meaning 'leg'. So, a triangle having two equal legs is called an isosceles triangle.

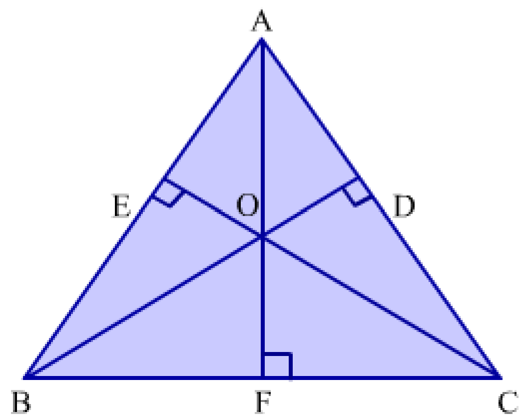
Know More

- The altitude to the base of an isosceles triangle bisects the vertex angle.
- The altitude to the base of an isosceles triangle bisects the base.
- When the equal sides are at right angle, the triangle is called a 'right isosceles triangle'.

Whiz Kid

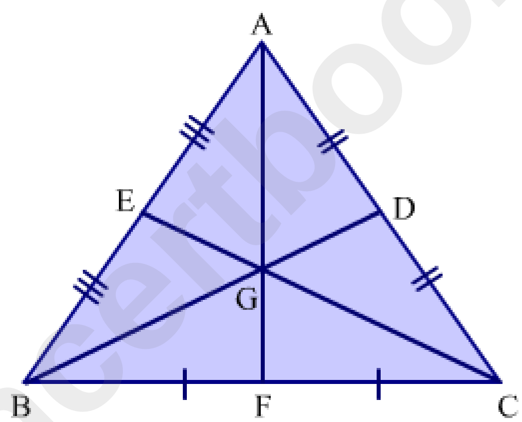
In an isosceles triangle, the orthocentre, the centroid, the incentre and the circumcentre, all lie on the median to the base.

Orthocentre: It is the point where the altitudes of the three sides of a triangle intersect.



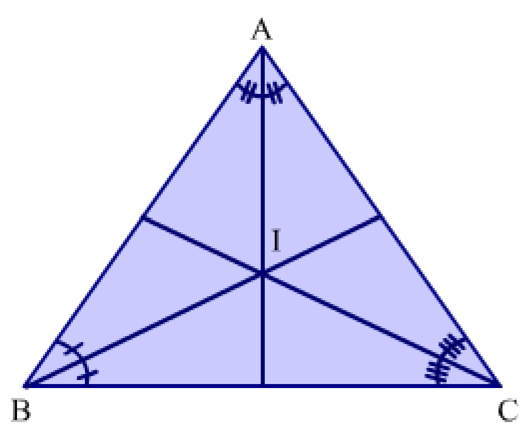
In ΔABC, O is the orthocentre.

Centroid: It is the point where the medians of the three sides of a triangle intersect.



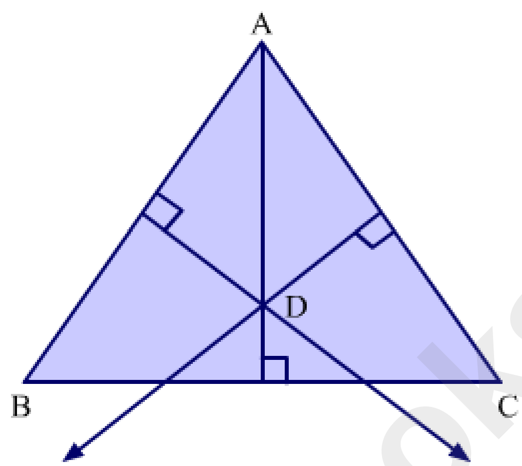
In ΔABC, G is the centroid.

Incentre: It is the point where the interior angle bisectors of a triangle intersect.



In ΔABC, I is the incentre.

Circumcentre: It is the point where the perpendicular bisectors of the three sides of a triangle intersect.



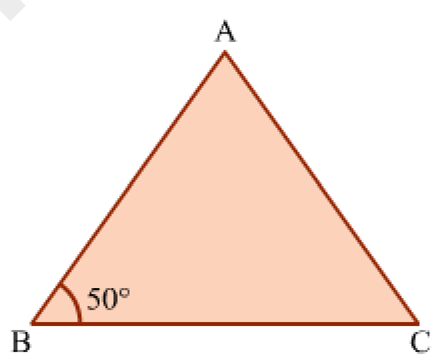
In ΔABC, D is the circumcentre.

Solved Examples

Easy

Example 1:

In the given ΔABC, AB = AC. What is the measure of ∠BAC?



Solution:

ΔABC is an isosceles triangle with AB = AC.

By the property of isosceles triangles, we obtain:

∠ABC = ∠ACB

⇒ ∠ACB = 50°

By the angle sum property, we have:

∠ABC + ∠ACB + ∠BAC = 180°

⇒ 50° + 50° + ∠BAC = 180°

⇒ ∠BAC = 180° − 100°

⇒ ∠BAC = 80°

Example 2:

Ram lives in a triangular tree house. The measure of each base angle is the same and the length of the base is 4 m. He wants to cover the slant sides with metal sheets. If the boundary of the triangular structure is 10 m, then what length of sheet does Ram need to cover the slant sides (irrespective of the width of the sheet).

Solution:

It is given that the base angles are equal; so, the triangular structure is isosceles. Hence, the slant sides are equal.

Now, we know that:

Base = 4 m

Boundary of the triangular structure = Perimeter of the isosceles triangle = 10 m

⇒ 10 m = 4 m + 2 × (Measure of each slant side)

⇒ 2 × (Measure of each slant side) = 10 m − 4 m

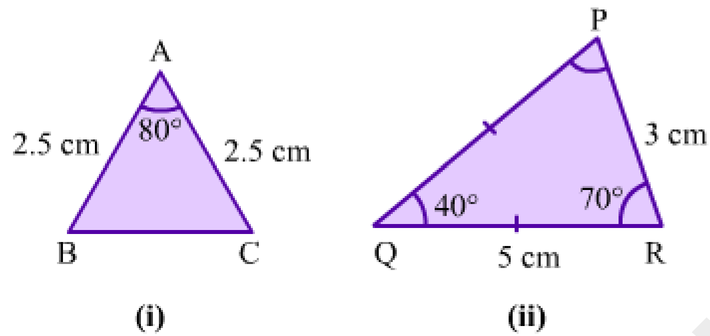
⇒ 2 × (Measure of each slant side) = 6 m

Hence, Ram needs 6 m of metal sheet to cover the slant sides of the triangular structure.

Medium

Example 1:

Find the missing angles in the following triangles.



Solution:

1. In ΔABC, AB = AC = 2.5 cm

   Since the angles opposite equal sides of a triangle are equal, we obtain:

   ∠ABC = ∠ACB

   Let ∠ABC = ∠ACB = x

   By the angle sum property of triangles, we have:

   ∠ABC + ∠ACB + ∠BAC = 180°

   ⇒ $x + x + 80° = 180°$

   ⇒ $2x = 100°$

   ⇒ $x = 50°$

   Thus, ∠ABC = ∠ACB = 50°

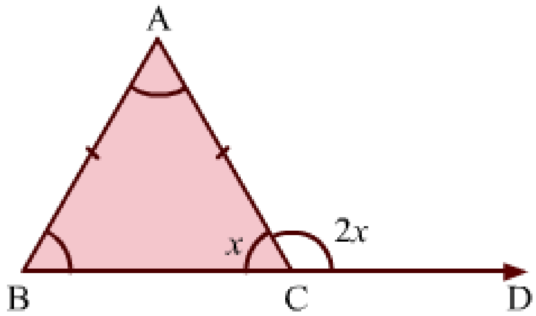
2. In ΔPQR, PQ = QR = 5 cm and PR = 3 cm

   Since PQ = QR, we obtain

   ∠QPR = ∠PRQ = 70°

Example 2:

The shown ΔABC is isosceles with AB = AC. Find the measures of ∠BAC, ∠ABC and ∠ACB.



Solution:

∠ACB and exterior angle ∠ACD form a linear pair.

∴ ∠ACB + ∠ACD = 180°

⇒ $x + 2x = 180°$

⇒ $3x = 180°$

⇒ ∴ $x = 60°$

So, ∠ACB = 60° and ∠ACD = 120°

ΔABC is isosceles.

∴ ∠ABC = ∠ACB = 60° (∵ Angles opposite equal sides are equal)

Now, by the exterior angle property, we have:

∠ACD = ∠BAC + ∠ABC

⇒ 120° = ∠BAC + 60°

⇒ ∠BAC = 60°

Thus, ∠BAC = ∠ABC = ∠ACB = 60°.

Example 3:

Prove that in an isosceles triangle, the angle bisector of the apex angle is the perpendicular bisector of the base.

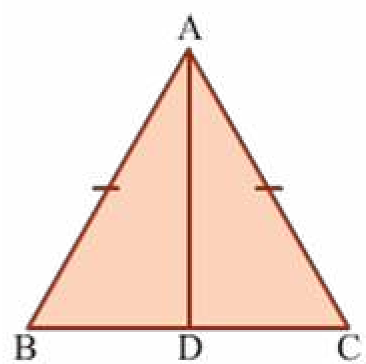
Solution:

Given: ABC is a triangle in which AB = AC and apex angle ∠A.

To Prove: AD is perpendicular bisector of BC and BD = DC.

Construction: Draw an angle bisector AD from A on BC.

Proof:



In ΔABD and ΔACD,
AB = *AC* (Given)

AD = *AD* (Common side)
∠BAD = ∠CAD (AD is a bisector of ∠A)

So, by SAS congruence criterion,
Δ ADB ≅ Δ ADC
⇒BD = DC and ∠ADB =∠ADC (CPCT)

Now,
∠ ADB + ∠ADC= 180° (Linear Pair)
⇒∠ ADB + ∠ ADB = 180°
⇒ 2∠ ADB = 180°

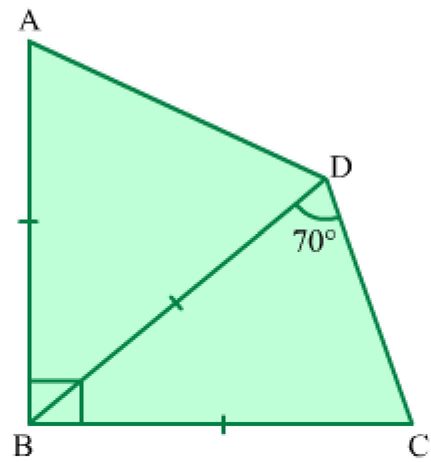
$$\Rightarrow \angle ADB = \frac{180^\circ}{2} = 90^\circ$$

Thus, AD is the perpendicular bisector of BC.

Hard

Example 1:

ΔABD is isosceles (figure not drawn to scale) with AB = BD, while ΔBCD is isosceles with BC = BD. Also, ∠BDC measures 70° and ∠ABC measures 90°. What is the measure of ∠ADC?



Solution:

In ΔBCD, BC = BD.

∴ ∠BCD = ∠BDC = 70º (∵ Angles opposite equal sides are equal)

In ΔBCD, by the angle sum property, we obtain:

∠BCD + ∠BDC + ∠CBD = 180º

⇒ 70º + 70º + ∠CBD = 180º

⇒ 140º + ∠CBD = 180º

⇒ ∠CBD = 40º

It is given that ∠ABC = 90º.

⇒ ∠ABD + ∠CBD = 90º

⇒ ∠ABD + 40º = 90º

⇒ ∠ABD = 50º

In ΔABD, we have:

AB = BD

∴ ∠BAD = ∠BDA = x (∵ Angles opposite equal sides are equal)

In ΔABD, by the angle sum property, we obtain:

$\angle ABD + \angle BAD + \angle BDA = 180^\circ$

$\Rightarrow 50^\circ + x + x = 180^\circ$

$\Rightarrow 2x = 130^\circ$

$\Rightarrow x = 65^\circ$

So, $\angle BAD = \angle BDA = 65^\circ$

Now,

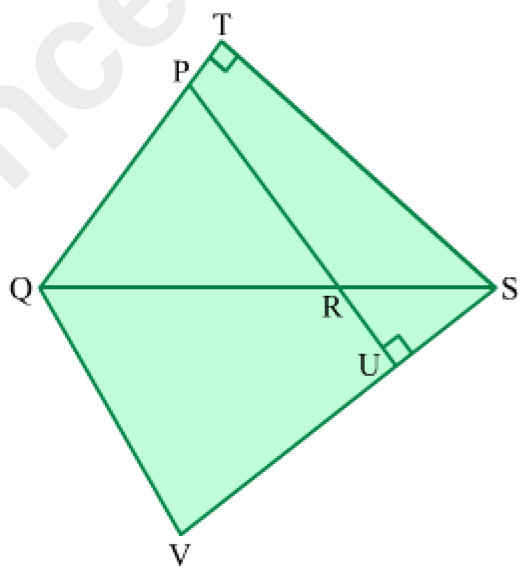
$\angle ADC = \angle BDA + \angle CDB$

$\Rightarrow \angle ADC = 65^\circ + 70^\circ$

$\Rightarrow \angle ADC = 135^\circ$

Example 2:

In the given figure, ΔPQR is isosceles with PQ = PR. Side SU is extended up to point V such that ST = SV. If $\angle QTS = \angle RUS = 90^\circ$, prove that QS bisects $\angle TSU$ and hence show that $\Delta QTS \cong \Delta QVS$.



Solution:

In ΔPQR, we have:

PQ = PR (Given)

∴ ∠PQR = ∠PRQ (∵ Angles opposite equal sides are equal)

It is given that ∠QTS = 90°.

Using the angle sum property in ΔQTS, we obtain:

∠TQS + ∠TSQ + ∠QTS = 180°

⇒ ∠TQS + ∠TSQ + 90° = 180°

⇒ ∠TSQ = 90° − ∠TQS … (1)

It is given that ∠RUS = 90°.

Using the angle sum property in ΔRUS, we obtain:

∠SRU + ∠RUS + ∠RSU = 180°

⇒ ∠SRU + 90° + ∠RSU = 180°

⇒ ∠RSU = 90° − ∠SRU … (2)

∠SRU = ∠PRQ (Vertically opposite angles)

Also, ∠PQR = ∠PRQ

∴ ∠SRU = ∠PQR … (3)

Now, from equations (2) and (3), we get:

∠RSU = 90° − ∠PQR

⇒ ∠RSU = 90° − ∠TQS … (4) [∵ ∠PQR = ∠TQS]

From equations (1) and (4), we get:

∠TSQ = ∠RSU

Hence, QS bisects ∠TSU.

Now, consider ΔQTS and ΔQVS.

QS = QS (Common side)

∠TSQ = ∠QSV (∵ ∠TSQ = ∠RSU and ∠RSU = ∠QSV)

ST = SV (Given)

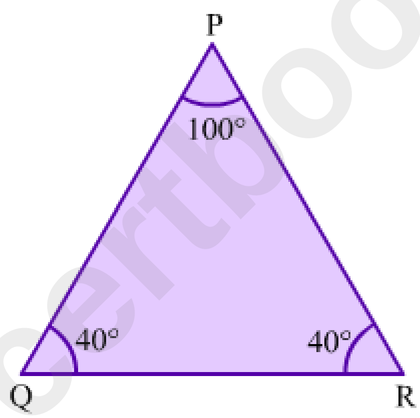
Thus, by the SAS congruency rule, we get:

ΔQTS ≅ ΔQVS

Sides Opposite to Equal Angles of a Triangle are Equal

Observing the Equal Angles and the Sides Opposite to Them in an Isosceles Triangle

Consider the following ΔPQR.



Is ΔPQR isosceles? We know that if two sides of a triangle are equal (or congruent), then the triangle is isosceles. However, in ΔPQR, two angles are equal (or congruent). We have studied that the angles opposite to equal (or congruent) sides of an isosceles triangle are equal (or congruent). Is the converse of this property also true?

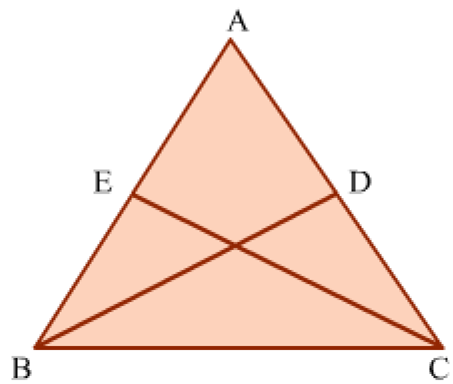
In this lesson, we will study about the equality of sides opposite equal angles in an isosceles triangle. We will also solve some examples related to this concept.

Sides Opposite To Equal Angles of a Triangle Are Equal

Whiz Kid

In an isosceles triangle, the medians drawn from the base vertices to the opposite sides are of equal length**.**

For example:



The shown ΔABC is isosceles such that AB = AC. BD and CE are the respective medians from vertices B and C to sides AC and AB. Therefore, BD = CE.

Solved Examples

Easy

Example 1:

In a ΔABC, ∠BAC = 2*x* and ∠ABC = ∠ACB = *x*. Find the value of *x* and hence show that AB = AC.

Solution:

It is given that ∠BAC = 2*x* and ∠ABC = ∠ACB = *x*.

By applying the angle sum property, we obtain:

∠BAC + ∠ABC + ∠ACB = 180°

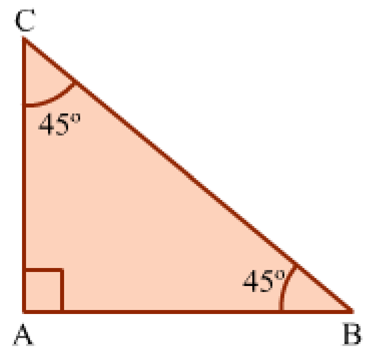
$\Rightarrow 2x + x + x = 180°$

$\Rightarrow 4x = 180°$

$\Rightarrow x = 45°$

So, ∠BAC = 2*x* = 2 × 45° = 90° and ∠ABC = ∠ACB = *x* = 45°

The given triangle can be drawn as is shown.



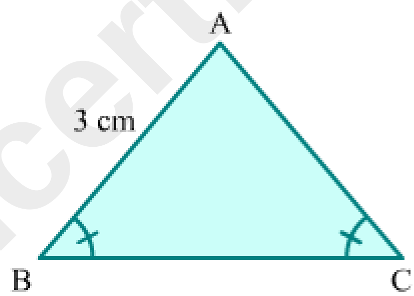
We know that the sides opposite equal angles of a triangle are equal.

∴ AB = AC

Hence, ΔABC is isosceles with AB = AC.

Example 2:

In the given ΔABC, ∠ABC = ∠ACB and the perimeter is 11 cm. Find the length of the base of the triangle.



Solution:

It is given that ∠ABC = ∠ACB and AB = 3 cm.

We know that the sides opposite equal angles of a triangle are equal.

∴ AC = AB = 3 cm

Perimeter of ΔABC = AB + BC + AC

⇒ 11 cm = 3 cm + BC + 3 cm

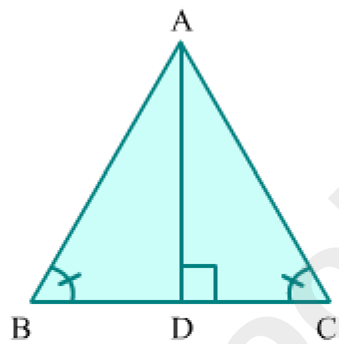
⇒ BC = 5 cm

Thus, the length of the base of ΔABC is 5 cm.

Medium

Example 1:

In the given ΔABC, ∠ABD = ∠ACD and AD is perpendicular to BC. Prove that AD bisects ∠BAC.



Solution:

In ΔABD and ΔACD, we have:

∠ABD = ∠ACD (Given)

∠ADB = ∠ADC = 90°

AB = AC (∵ Sides opposite equal angles are equal)

Thus, by the AAS congruence rule, we have:

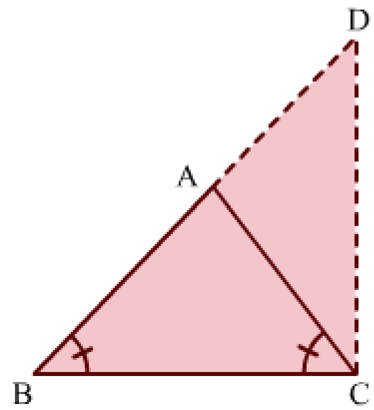
ΔABD ≅ ΔACD

∴ ∠BAD = ∠CAD (By CPCT)

Thus, AD bisects ∠BAC.

Example 2:

In the given ΔABC, ∠ABC = ∠ACB. Side BA is produced up to point D such that AB = AD. Prove that ∠BCD is a right angle.



Solution:

In ΔABC, we have:

∠ABC = ∠ACB … (1)   [Given]

∴ AB = AC   (∵ Sides opposite equal angles are equal)

Now,

AB = AD   (Given)

AD = AC   (∵ AB = AC)

Thus, in ΔADC, we have:

∠ACD = ∠ADC … (2)   (∵ Angles opposite equal sides are equal)

On adding equations (1) and (2), we get:

∠ACB + ∠ACD = ∠ABC + ∠ADC

⇒ ∠BCD = ∠ABC + ∠BDC   (∵ ∠ADC = ∠BDC)

On adding ∠BCD to both sides of the equation, we get:

∠BCD + ∠BCD = ∠ABC + ∠BDC + ∠BCD

⇒ 2∠BCD = 180°   (By the angle sum property)

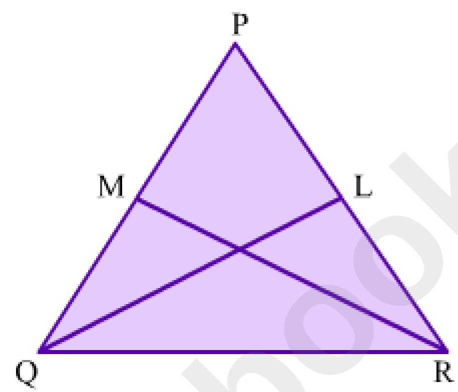
⇒ ∠BCD = 90°

Hence, ∠BCD is a right angle.

Hard

Example 1:

The shown ΔPQR is isosceles with PQ = PR. QL and RM are the respective medians from vertices Q and R to sides PR and PQ. Prove that the medians have the same length.



Solution:

It is given that PQ = PR.

Also, QL and RM are medians.

∴ PL = LR and PM = MQ

So,

PL + LR = PR

LR + LR = PR

2LR = PR

LR = $\frac{PR}{2}$ … (1)

Similarly, MQ = $\frac{PQ}{2}$ … (2)

Since PQ = PR, using equations (1) and (2), we obtain:

LR = MQ … (3)

In ΔQRL and ΔRQM, we have:

LR = MQ (From (3))

∠LRQ = ∠MQR (∵ Angles opposite equal sides are equal)

QR = RQ (Common side)

∴ ΔQRL ≅ ΔRQM (By the SAS congruence criterion)

⇒ QL = RM (By CPCT)

Thus, the medians QL and RM have the same length.

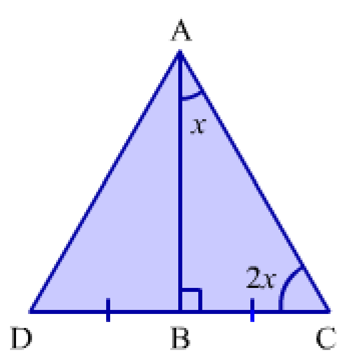
Example 2:

In a right-angled triangle ABC, ∠ACB = 2∠BAC. Prove that AC = 2BC.

Solution:

The given ΔABC can be drawn as is shown.

Construction: Produce CB up to point D such that BD = BC. Join point A to point D.



In ΔABD and ΔABC, we have:

BD = BC (By construction)

∠ABD = ∠ABC = 90°

AB = AB (Common side)

So, by the SAS congruence rule, we obtain:

$\Delta ABD \cong \Delta ABC$

⇒ AD = AC and ∠DAB = ∠BAC (By CPCT)

Let ∠BAC be x. Then, ∠DAB will also be x.

Now, ∠DAC = ∠DAB + ∠BAC

⇒ ∠DAC = $x + x$

⇒ ∠DAC = $2x$

⇒ ∠DAC = ∠ACB (∵ ∠ACB = 2∠BAC = $2x$)

⇒ DC = AD (∵ Sides opposite equal angles are equal)

Since BC = DB, we have:

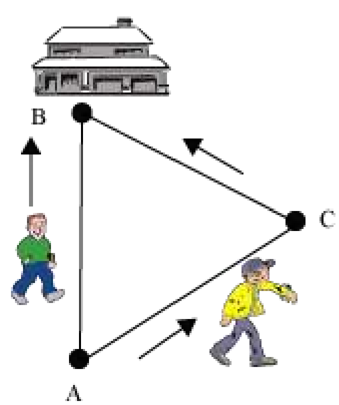
DC = 2BC

⇒ 2BC = AD

⇒ 2BC = AC (∵ We have proved AD = AC)

Triangle Inequalities

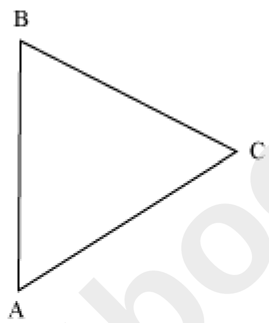
Rohit and Mohit are brothers. Their house is located at position B, as shown in the following figure.



They are standing at a position A and want to reach their house. Rohit chooses the path AB to reach his house while Mohit first goes to position C starting from A and then travels distance CB to reach his house. Both of them walk at the same speed. **Who will take less time to reach the house?**

Since their speeds are the same, more time will be taken by the person who has to cover the larger distance. Therefore, we can say that Rohit will take less time in reaching the house as the path followed by Rohit is shorter than the path followed by Mohit.

We can see that the paths followed by Rohit and Mohit form a triangle.



i.e. ABC is a triangle. From the above example, we can clearly see that length AB is shorter than length BC + CA. Or, we can say that length BC + CA is longer than length AB.

Therefore, we can conclude that

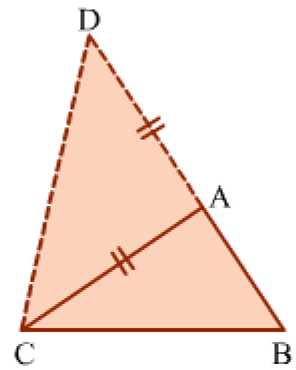
> “**The sum of the lengths of any two sides of a triangle is always greater than the length of the third side of the triangle**”.

Now, from the above discussion, we can say that in ΔABC, AB + BC > CA, AB + CA > BC and CA + BC > AB.

Let us go through the proof of the above property.

Proof:

Let us take any ΔABC and extend the side AB to D such that AD = CA.



In ΔACD, we have

AD = CA

⇒ ∠ACD = ∠ADC

⇒ ∠ACD + ∠ACB > ∠ADC

⇒ ∠BCD > ∠BDC (∠BDC is same as ∠ADC)

⇒ BD > BC (Side opposite to greater angle is longer)

⇒ AB + AD > BC

⇒ AB + CA > BC (AD = CA)

Similarly, we can prove that AB + BC > CA and CA + BC > AB.

From the result CA + BC > AB, we can deduce that CA − AB < BC. Similarly, we can deduce results from the remaining two inequalities.

Thus, from this, we can conclude that

"The difference between the lengths of any two sides of a triangle is always smaller than the length of the third side of the triangle".

The inequalities AB + CA > BC, AB + BC > CA and CA + BC > AB are called triangle inequalities. They are necessary condition for the existence of a triangle with sides AB, BC and CA. This tells that the straight line is the shortest path between any two points. Given three numbers a, b, c, then the necessary condition for the existence of a triangle with sides a, b, c are that $b + c > a$, $c + a > b$ and $a + b > c$.

Let us now look at some examples to understand this property of triangles better.

Example 1:

Check whether it is possible to have a triangle with the following sides or not.

1. **11.5 cm, 7.8 cm, 14.7 cm**
2. **7.4 cm, 18.5 cm, 10.9 cm**

Solution:

1. 11.5 cm, 7.8 cm, 14.7 cm

   For a triangle, the sum of the lengths of any two sides must be greater than the third side.

   Now,

   11.5 cm + 7.8 cm = 19.3 cm > 14.7 cm

   11.5 cm + 14.7 cm = 26.2 cm > 7.8 cm

   7.8 cm + 14.7 cm = 22.5 cm > 11.5 cm

   We can see that the sum of the lengths of any two sides is greater than the length of the third side.

   Therefore, a triangle with sides of given lengths is possible.

2. 7.4 cm, 18.5 cm, 10.9 cm

   For a triangle, the sum of the lengths of any two sides must be greater than the third side.

   But in this case,

   7.4 cm + 10.9 cm = 18.3 cm < 18.5 cm

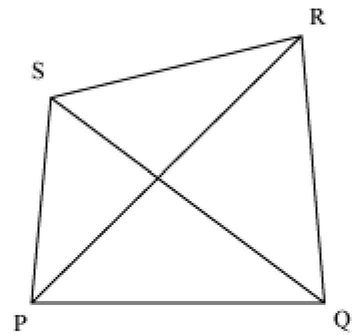
   Therefore, a triangle with sides of given lengths is not possible.

Example 2:

Can the sum of the four sides of a quadrilateral be equal to the sum of the lengths of its diagonals?

Solution:

Consider a quadrilateral PQRS in which PR and QS are its two diagonals.



Now, we know that the sum of any two sides of a triangle is always greater than the third side. Therefore,

In ΔPQR,

PQ + QR > PR … (1)

In ΔPRS,

PS + RS > PR … (2)

In ΔPQS,

PQ + PS > QS … (3)

In ΔSRQ,

QR + RS > QS … (4)

On adding (1), (2), (3), and (4), we obtain

PQ + QR + PS + RS + PQ + PS + QR + RS > PR + PR + QS + QS

⇒ (PQ + PQ) + (QR + QR) + (RS + RS) + (PS + PS) > (PR + PR) + (QS + QS)

⇒ 2PQ + 2QR + 2RS + 2PS > 2PR + 2QS

⇒ 2(PQ + QR + RS + PS) > 2 (PR + QS)

On dividing both sides by 2, we obtain

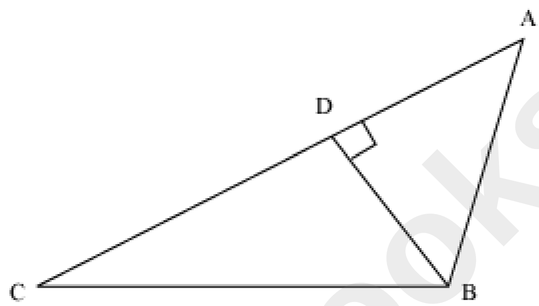
PQ + QR + RS + PS > PR + QS

Therefore, the sum of the lengths of the four sides of a quadrilateral is always greater than the sum of the lengths of its diagonals.

Hence, the sum of the lengths of the four sides cannot be equal to the sum of the lengths of its two diagonals.

Example 3:

In the given figure, BD ⊥ AC.



Which of the following statements is correct?

(i) 2 BD > AB + BC + CA

(ii) 2 BD < AB + BC + CA

Solution:

We know that the sum of any two sides of a triangle is always greater than the third side.

Therefore, in ΔABD DA + AB > BD … (i)

In ΔBCD,

BC + CD > BD … (ii)

On adding (i) and (ii), we obtain

AB + DA + BC + CD > BD + BD

⇒ AB + BC + (CD + DA) > 2 BD

Now, from the figure,

CD + DA = CA

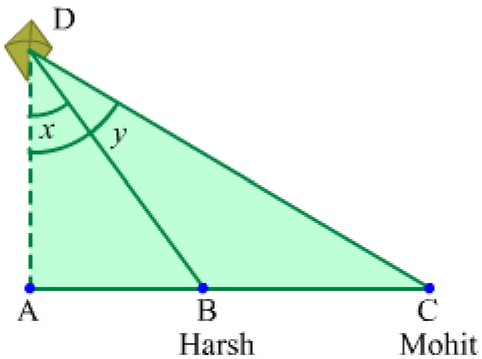
Therefore,

AB + BC + CA > 2 BD

i.e. 2 BD < AB + BC + CA

Therefore, statement **(ii)** is correct.

Observing the Relation between the Angles and Sides of a Triangle



Harsh is flying a kite. The thread of the kite makes angle *x* with the vertical, as is shown in the figure. Sometime later, he gives the thread to his friend Mohit who is standing some distance away from him. At that position, the thread makes angle *y* with the vertical.

The figure clearly shows that angle *y* of ΔACD is greater than angle *x* of ΔABD. Also, side AC is greater than side AB. This shows us that if we increase the length of any side of a triangle, then the angle facing that side also increases.

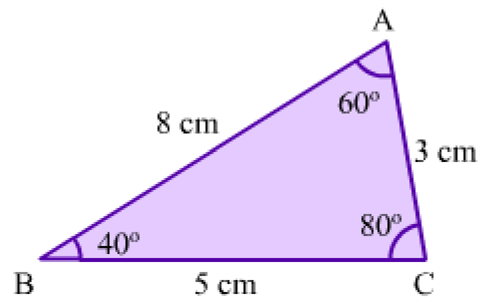
In this lesson, we will discuss the relation between the angles and sides of a triangle. We will then solve some examples relating to the same.

Triangle Inequality Theorem

We know that if two sides of a triangle are equal, then the angles opposite these sides are also equal. Now, what **if all the sides of a triangle are unequal? What can be said about its angles?** The **triangle inequality theorem** describes such a triangle. It states that:

If two sides of a triangle are unequal, then the longer side has the greater angle opposite it.

Consider the given ΔABC.



Let us apply the stated theorem in this triangle.

AB = 8 cm is the longest side in ΔABC. Therefore, the angle opposite AB, i.e., ∠BCA is the greatest angle of the triangle. Also, AC = 3 cm is the shortest side in ΔABC. Therefore, the angle opposite AC, i.e., ∠ABC is the smallest angle of the triangle.

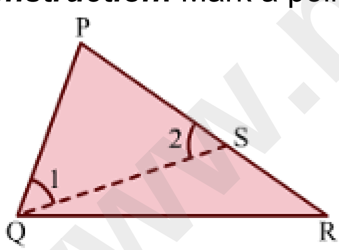
Thus, we can conclude that the angle opposite the longer side is greater. Therefore, the theorem holds true.

Proving the Triangle Inequality Theorem

Given: ΔPQR in which PR > PQ

To prove: ∠Q > ∠R

Construction: Mark a point S on PR such that PQ = PS. Join Q to

S.  PP

Proof: In ΔPQS, we have:

PQ = PS (By construction)

⇒ ∠1 = ∠2 ... (1) [∵ Angles opposite equal sides are equal]

In ΔQRS, ∠2 is the exterior angle; so, it is greater than the interior opposite angles of ΔQRS.

∴ ∠2 > ∠SRQ

⇒ ∠2 > ∠PRQ … (2) [∵ ∠PRQ = ∠SRQ]

From (1) and (2), we have:

∠1 > ∠PRQ ... (3)

Now, ∠1 is a part of ∠PQR.

So, ∠PQR > ∠1 … (4)

Thus, from (3) and (4), we can conclude that:

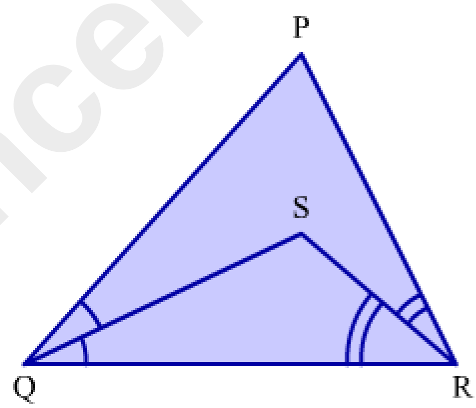
∠PQR > ∠PRQ

Solved Examples

Easy

Example 1:

In the given ΔPQR, PQ is greater than PR. Also, QS and RS are the respective bisectors of ∠PQR and ∠PRQ. Prove that ∠SRQ > ∠SQR.



Solution:

In ΔPQR, we have:

PQ > PR (Given)

⇒ ∠PRQ > ∠PQR (∵ Angle opposite longer side is greater)

$$\Rightarrow \frac{\angle PRQ}{2} > \frac{\angle PQR}{2}$$

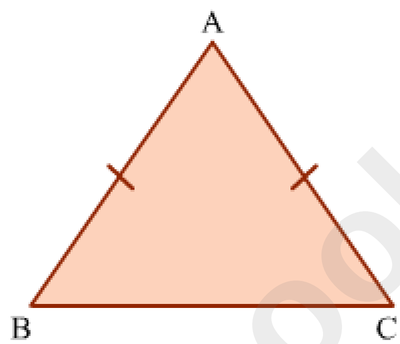
⇒ ∠SRQ > ∠SQR (∵ RS bisects ∠PRQ and QS bisects ∠PQR)

Example 2:

ABC is an isosceles triangle with AB = AC and AB < BC. Prove that ∠BAC > ∠ABC.

Solution:

Consider the following ΔABC in which AB = AC and AB < BC.



In ΔABC, we have:

∠ABC = ∠ACB … (1) [∵ Angles opposite equal sides AB and AC are equal]

By the triangle inequality theorem, we have:

∠ACB < ∠BAC … (2) [∵ Angle opposite longer side BC is greater]

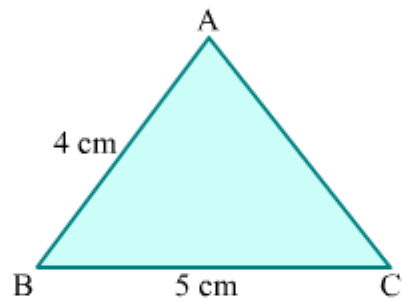
Thus, from (1) and (2), we obtain:

∠BAC > ∠ABC

Medium

Example 1:

In the shown ΔABC, AB = 4 cm, BC = 5 cm and the perimeter is 16 cm. Determine the smallest and greatest angles of the triangle.



Solution:

In ΔABC, we have:

AB = 4 cm and BC = 5 cm (Given)

Perimeter = 16 cm (Also given)

⇒ AB + BC + CA = 16 cm (∵ Perimeter is the sum of all sides)

⇒ 4 cm + 5 cm + CA = 16 cm

⇒ CA = 7 cm

Now, CA = 7 cm is the longest side of ΔABC. Thus, the angle opposite it, i.e., ∠ABC is the greatest angle of the triangle.

∴ ∠ABC > ∠BAC and ∠ABC > ∠ACB … (1)

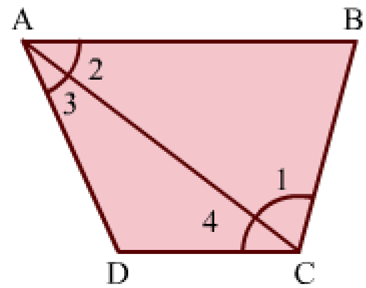
Also, BC > AB

∴ ∠BAC > ∠ACB … (2)

Thus, from (1) and (2), we can conclude that ∠ABC is the greatest angle and ∠ACB is the smallest angle in ΔABC.

Example 2:

Suppose AB is the longest side and CD the shortest side of the given quadrilateral ABCD. Then, prove that ∠BCD > ∠BAD.



Solution:

In ΔABC, AB is the longest side.

So, AB > BC

⇒ ∠1 > ∠2 … (1)   [∵ Angle opposite longer side is greater]

In ΔADC, CD is the shortest side.

So, AD > CD

⇒ ∠4 > ∠3 … (2)   [∵ Angle opposite longer side is greater]

On adding (1) and (2), we get:

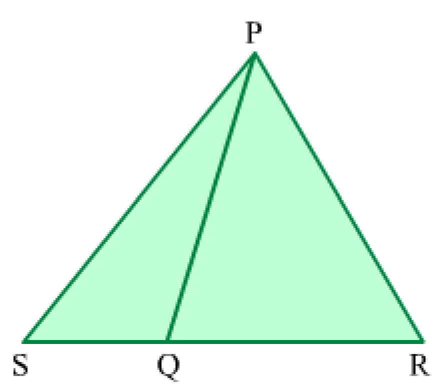
∠1 + ∠4 > ∠2 + ∠3

⇒ ∠BCD > ∠BAD

Hard

Example 1:

In the given figure, PQ = PR. Show that ∠PRS > ∠PSR.



Solution:

In ΔPQR, we have:

PQ = PR

⇒ ∠PRQ = ∠PQR … (1)   [∵ Angles opposite equal sides are equal]

In ΔPSQ, SQ is produced to R.

So, ∠PQR = ∠PSQ + ∠SPQ   (By the exterior angle property)

⇒ ∠PQR > ∠PSQ … (2)

From (1) and (2), we obtain:

∠PRQ >∠PSQ

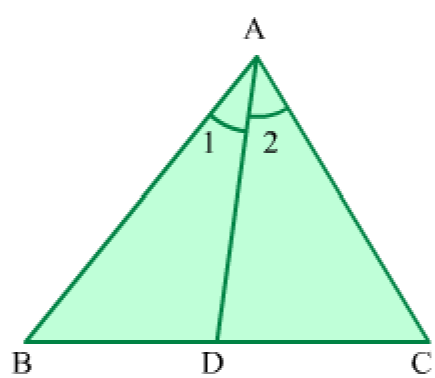
⇒ ∠PRS > ∠PSR (∵ ∠PRQ = ∠PRS and ∠PSQ = ∠PSR)

Example 2:

In a ΔABC, AB > AC and AD is the bisector of ∠BAC. Show that ∠ADB > ∠ADC.

Solution:

Let the following ΔABC be the given triangle such that AB > AC. Also, AD is the bisector of ∠BAC.



In ΔABC, we have:

AB > AC (Given)

⇒ ∠ACB > ∠ABC   (∵ Angle opposite longer side is greater)

On adding ∠1 to both sides, we get:

∠ACB + ∠1 > ∠ABC + ∠1

⇒ ∠ACB + ∠2 > ∠ABC + ∠1 … (1)   [∵ AD bisects ∠BAC; ∠1 = ∠2]

By the exterior angle property, we have:

∠ADB = ∠ACB + ∠2 … (2)

Similarly, ∠ADC = ∠ABC + ∠1 … (3)

Thus, by using (1), (2) and (3), we can conclude that:

∠ADB > ∠ADC

Observation of the Sides of a Triangle by Seeing the Angles

Consider the following triangular racetrack where two cars start from different points A and B to reach the finish point C.

The red car has to travel at an angle of 65°, while the blue car has to travel at an angle of 50° with respect to line AB to reach the finish point C. Do you think any one car has an advantage over the other?

On observing the triangular track, it seems that path BC is longer than path AC. So, clearly, the red car has an advantage over the blue car. Also, the angle opposite BC is greater than the angle opposite AC. So, what does this tell us about the relation between the sides and angles of the triangular racetrack?

Let us go through this lesson to learn about the relation between the sides and angles of a triangle. We will also solve some problems based on this relation.

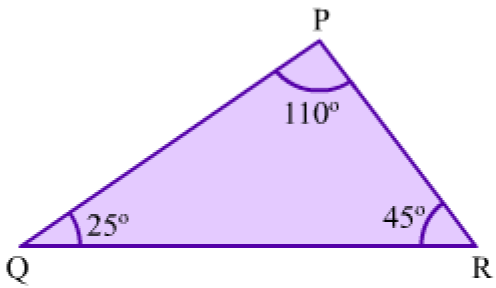
In a Triangle, the Side Opposite the Greater Angle is

Longer

We have studied that in a triangle having two unequal sides, the angle opposite the longer side is greater. The converse of this property is also true. It states that:

If two angles of a triangle are unequal, then the greater angle has the longer side opposite it. In other words, the smaller angle has the shorter side opposite it.

Consider the following ΔPQR.



Let us apply the stated property in this triangle.

∠QPR = 110° is the greatest angle in ΔPQR.

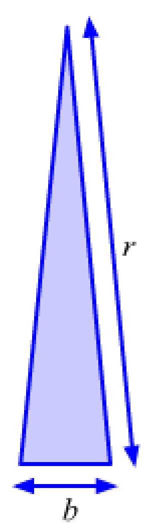
Therefore, the side opposite ∠QPR, i.e., QR is the longest side of the triangle. Also, ∠PQR = 25° is the smallest angle in ΔPQR.

Therefore, the side opposite ∠PQR, i.e., PR is the shortest side of the triangle.

We can apply this property to other triangles as well. Using this property, we can say that in a right triangle, the hypotenuse is the longest side.

Whiz Kid

In trigonometry, a skinny triangle is one whose height is much greater than its base.



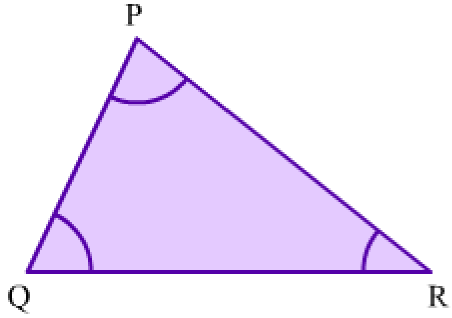
The given triangle is an example of a skinny triangle.

Proving the Property

Given: ΔPQR in which ∠Q > ∠R

To prove: PR > PQ.

Proof: In ΔPQR, we can have three possible cases: (1) PQ > PR, (2) PQ = PR and (3) PQ < PR.



CASE 1:

When PQ > PR, we have:

∠R > ∠Q (∵ Angle opposite longer side is greater)

But this contradicts the given hypothesis that ∠R < ∠Q. Thus, PQ > PR is not true.

CASE 2:

When PQ = PR, we have:

∠R = ∠Q (∵ Angles opposite equal sides are equal)

But this too contradicts the given hypothesis that ∠R < ∠Q. Thus, PQ = PR is also not true.

So, we are left with the third possibility, i.e., PQ < PR (or PR > PQ), which must be true.

Thus, we have proven that if two angles of a triangle are unequal, then the greater angle has the longer side opposite it.

Solved Examples

Easy

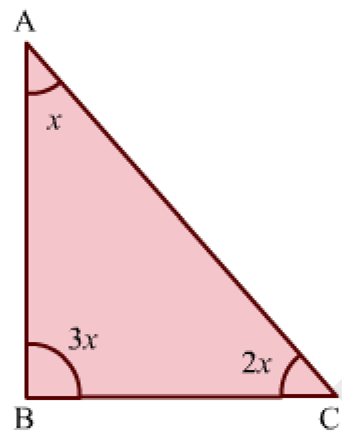
Example 1:

The angles of a triangle are in the ratio 1 : 2 : 3. Find the greatest angle and identify the longest side of the triangle.

Solution:

It is given that the angles of the triangle, say ΔABC, are in ratio 1 : 2 : 3.

Let the angles be x, $2x$ and $3x$, as is shown in the figure.



By the angle sum property of triangles, we have:

$\angle BAC + \angle ABC + \angle ACB = 180°$

$\Rightarrow x + 3x + 2x = 180°$

$\Rightarrow 6x = 180°$

$\Rightarrow x = 30°$

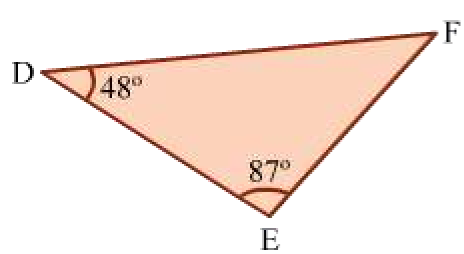
So, $2x = 2 \times 30° = 60°$ and $3x = 3 \times 30° = 90°$

Thus, the greatest angle in the given triangle is ∠ABC, i.e., 90°.

Now, we know that the side opposite the greater angle is longer. In ΔABC, side AC is opposite the greatest angle; hence, it is the longest.

Example 2:

Which side of the given triangle is the shortest?



Solution:

In order to find the shortest side of ΔDEF, we need to figure out the smallest angle of the triangle. This is because the smallest side is opposite the smallest angle.

By the angle sum property of triangles, we have:

$\angle EDF + \angle DEF + \angle EFD = 180°$

$\Rightarrow 48° + 87° + \angle EFD = 180°$

$\Rightarrow 135° + \angle EFD = 180°$

$\Rightarrow \angle EFD = 45°$

Clearly, ∠EFD has the smallest measure in the given triangle. So, the side opposite it, i.e., DE is the shortest side of the triangle.

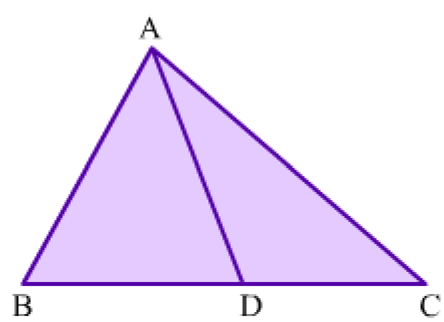
Medium

Example 1:

In the given ΔABC, ∠ABD > ∠ACD and AD is the bisector of ∠BAC.

Prove that:

1. AB < AC
2. ∠ADB < ∠ADC



Solution:

1. It is given that ∠ABD > ∠ACD.

   ∴ AC > AB (∵ Side opposite greater angle is longer)

Or, AB < AC

2. In ΔABD, we have:

∠ABD + ∠BAD + ∠ADB = 180° (By the angle sum property)

⇒ ∠ACD + ∠BAD + ∠ADB < 180° (∵ ∠ABD > ∠ACD)

⇒ ∠ACD + ∠CAD + ∠ADB < 180° (∵ AD bisects ∠BAC; ∠BAD = ∠CAD)

⇒ ∠ADB < 180° − (∠ACD +∠CAD)

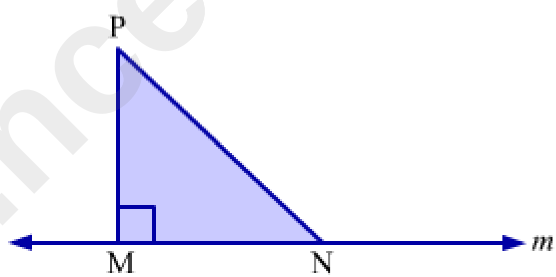
⇒ ∠ADB < ∠ADC (By the angle sum property in ΔADC)

Example 2:

Prove that of all the line segments that can be drawn to a given line from a point lying outside the line, the perpendicular line segment is the shortest.

Solution:

Let there be a straight line *m*, a point P lying outside the line and a point M lying on the line. Also, PM ⊥ *m* and N is any point other than M on *m*.



Since PM is perpendicular to *m*, ∠PMN = 90°.

In ΔPMN, we have:

⇒ ∠PMN + ∠MPN + ∠PNM = 180° (By the angle sum property)

⇒ 90° + ∠MPN + ∠PNM = 180°

⇒ ∠MPN + ∠PNM = 90°

⇒ ∠PNM < 90°

⇒ ∠PNM < ∠PMN

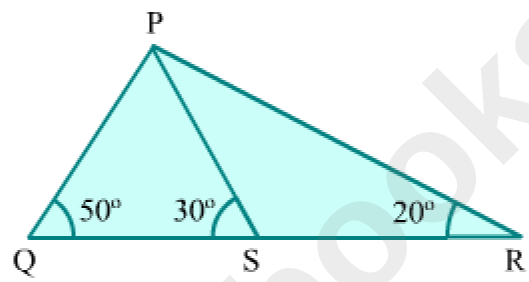
⇒ PM < PN (∵ Side opposite greater angle is longer)

Thus, PM is the shortest of all line segments from point P to line *m*.

Hard

Example 1:

In the given ΔPQR, ∠PQR = 50°, ∠PRQ = 20° and ∠PSQ = 30°. Prove that QS > SR.



Solution:

In ΔPQS, we have:

∠PQR = 50° and ∠PSQ = 30° (Given)

By using the angle sum property in ΔPQS, we obtain:

∠PQR + ∠PSQ + ∠QPS = 180°

⇒ 50° + 30° + ∠QPS = 180°

⇒ ∠QPS = 180° − (50° + 30°)

⇒ ∠QPS = 100°

So, ∠QPS > ∠PQR > ∠PSQ

Since the side opposite the greater angle is longer, we get:

QS > PS > PQ … (1)

∠PSQ and ∠PSR form a linear pair.

So, ∠PSQ + ∠PSR = 180°

⇒ 30° + ∠PSR = 180°

⇒ ∠PSR = 180° − 30°

⇒ ∠PSR = 150°

In ΔPSR, we have:

∠PSR + ∠SPR + ∠PRQ = 180° (By the angle sum property)

⇒ 150° + ∠SPR + 20° = 180°

⇒ ∠SPR = 180° − (150° + 20°)

⇒ ∠SPR = 10°

So, ∠PSR > ∠PRQ > ∠SPR

Since the side opposite the greater angle is longer, we get:

PR > PS > SR … (2)

By using (1) and (2), we can conclude that:

QS > SR

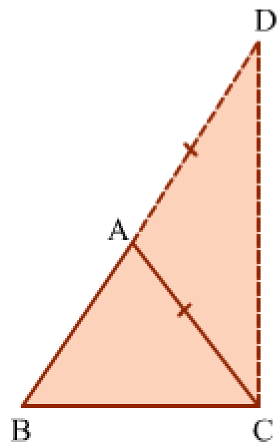
Example 2:

Prove that the difference between any two sides of a triangle is less than the third side.

Solution:

We have to prove that the difference between any two sides of a triangle (let us say ΔABC) is less than the third side of the triangle. We will do so by showing that in ΔABC, BC − AC < AB.

Construction**:** Extend side BA up to point D such that AD = AC. Join C to D.



In ΔACD, we have:

AD = AC … (1)

We know that in an isosceles triangle, the angles opposite equal sides are equal.

∴ ∠ACD = ∠ADC

⇒ ∠ACD + ∠ACB > ∠ADC

⇒ ∠BCD > ∠ADC

We know that the side opposite the greater angle is longer. So, we obtain, BD > BC

⇒ AB + AD > BC (∵ BD = AB + AD)

⇒ AB + AC > BC (Using equation 1)

⇒ BC − AC < AB

Similarly, we can prove that AB − BC < AC and AC − AB < BC.

Thus, we have proved that the difference between any two sides of a triangle is less than the third side.