Indices and Logarithms

Laws of Rational Exponents of Real Numbers

Exponents or Indices

The term 'exponent' refers to the number of times a quantity is multiplied with itself. It is also called 'index' or 'power'. The exponential form of any number is x^y, where **x is the base and y is the exponent**. It is read as 'y^{th} power of x' or 'x raised to the power y'.

Till now, we have studied the laws of exponents for numbers having non-zero integers as the base and the **integral exponent**. For example, consider the expression $11^5 \div 11^3$. It can be simplified using the law of exponents $a^p \div a^q = a^{(p-q)}$ as follows: $11^5 \div 11^3 = 11^{(5-3)} = 11^2$

But what about numbers whose base is any real number or whose exponent is any rational number?

Can we simplify them using the same laws? Go through this lesson to find out how to simplify such expressions.

Numbers with Fractional Indices

While studying about the exponents, we come across few numbers having fractional indices. For example, $8^{\frac{1}{3}}$, $16^{\frac{1}{2}}$, $81^{\frac{1}{4}}$ etc.

Do you know what these numbers mean?

We know that the square root of a number is represented with the index $\frac{1}{2}$ as well as with the symbol $\sqrt{\ }$.

For example, the square root of 16 can be represented as $16^{\frac{1}{2}}$ and $\sqrt{16}$.

Similarly, the square root of any real number a can be represented as $a^{\frac{1}{2}}$ and $\sqrt{a}$.

So, there are two ways to represent the square roots of numbers.

In the same manner, we can represent the **cube root, fourth root, fifth root,..., n^{th} root** of a real number with the symbol $\sqrt{\ }$ as well as with fractional indices.

For example, the cube root of 8 can be represented as $\sqrt[3]{8}$ and $8^{\frac{1}{3}}$. The fourth root of 81 can be represented as $\sqrt[4]{81}$ and $81^{\frac{1}{4}}$.

Similarly, the *n*th root of a number can also be represented as $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$.

Solved Examples

Easy

Example 1:

Find the values of the following index numbers.

i. $27^{\frac{1}{3}}$

ii. $256^{\frac{1}{4}}$

iii. $576^{\frac{1}{2}}$

iv. $4096^{\frac{1}{6}}$

Solution:

i. $27^{\frac{1}{3}}$ means the cube root of 27.

Therefore,

$$27^{\frac{1}{3}} = \sqrt[3]{27}$$
$$\Rightarrow 27^{\frac{1}{3}} = \sqrt[3]{3 \times 3 \times 3}$$
$$\Rightarrow 27^{\frac{1}{3}} = 3$$

ii. $256^{\frac{1}{4}}$ means fourth root of 256.

Therefore,

$$256^{\frac{1}{4}} = \sqrt[4]{256}$$
$$\Rightarrow 256^{\frac{1}{4}} = \sqrt[4]{4 \times 4 \times 4 \times 4}$$
$$\Rightarrow 256^{\frac{1}{4}} = 4$$

iii. $576^{\frac{1}{2}}$ means square root of 576.

Therefore,

$$576^{\frac{1}{2}} = \sqrt{576}$$
$$\Rightarrow 576^{\frac{1}{2}} = \sqrt{24 \times 24}$$
$$\Rightarrow 576^{\frac{1}{2}} = 24$$

iv. $4096^{\frac{1}{6}}$ means sixth root of 4096.

Therefore,

$$4096^{\frac{1}{6}} = \sqrt[6]{4096}$$
$$\Rightarrow 4096^{\frac{1}{6}} = \sqrt[6]{4 \times 4 \times 4 \times 4 \times 4 \times 4}$$
$$\Rightarrow 4096^{\frac{1}{6}} = 4$$

Laws of Exponents for Real Numbers

Consider two real numbers *a* and *b* and two rational numbers *m* and *n*. The laws of exponents involving these real bases and rational exponents can be written as follows:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$
- $(a^m)^n = a^{mn} = (a^n)^m$
- $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$
- $a^m \times b^m = (ab)^m$
- $a^{-m} = \frac{1}{a^m}$
- a0=1a0=1

Examples Based on the Laws of Exponents

Solved Examples

Easy

Example 1:

Simplify the following expressions.

1. $\left(\frac{27}{125}\right)^{\frac{2}{3}}$
2. $\sqrt[3]{(512)^{-2}}$

Solution:

i) $\left(\frac{27}{125}\right)^{\frac{2}{3}}$

$$= \left[\frac{(3)^3}{(5)^3}\right]^{\frac{2}{3}}$$

$$= \left[\left(\frac{3}{5}\right)^3\right]^{\frac{2}{3}} \qquad \left[\because \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m\right]$$

$$= \left(\frac{3}{5}\right)^{3\times\frac{2}{3}} \qquad \left[\because \left(a^m\right)^n = a^{m\times n}\right]$$

$$= \left(\frac{3}{5}\right)^2$$

$$= \frac{(3)^2}{(5)^2} \qquad \left[\because \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\right]$$

$$= \frac{9}{25}$$

ii) $\sqrt[3]{(512)^{-2}}$

$=\left[(512)^{-2}\right]^{\frac{1}{3}}$

$=(512)^{\frac{-2}{3}}$ $\left[\because (a^m)^n = a^{mn}\right]$

$=(8^3)^{\frac{-2}{3}}$

$=(8)^{3\times\frac{-2}{3}}$ $\left[\because (a^m)^n = a^{mn}\right]$

$=(8)^{-2}$

$=\frac{1}{8^2}$ $\left[\because a^{-m} = \frac{1}{a^m}\right]$

$=\frac{1}{64}$

Example 2:

Simplify the expression $\left(\sqrt{\frac{2}{3}}\right)^{\frac{3}{5}} \times \left(\sqrt{\frac{1}{7}}\right)^{\frac{3}{5}}$.

Solution:

$\left(\sqrt{\frac{2}{3}}\right)^{\frac{3}{5}} \times \left(\sqrt{\frac{1}{7}}\right)^{\frac{3}{5}}$

$=\left(\sqrt{\frac{2}{3}} \times \sqrt{\frac{1}{7}}\right)^{\frac{3}{5}}$ $[\because a^m \times b^m = (ab)^m]$

$=\left(\sqrt{\frac{2}{21}}\right)^{\frac{3}{5}}$

Example 3:

Simplify $\sqrt[3]{2^4} \times \sqrt[3]{3^4}$.

Solution:

$\sqrt[3]{2^4} \times \sqrt[3]{3^4}$

$= \left(2^4\right)^{\frac{1}{3}} \times \left(3^4\right)^{\frac{1}{3}}$

$= (2)^{\frac{4}{3}} \times (3)^{\frac{4}{3}} \quad \left[\because \left(a^m\right)^n = a^{mn}\right]$

$= (2 \times 3)^{\frac{4}{3}} \quad \left[\because a^m \times b^m = (ab)^m\right]$

$= 6^{\frac{4}{3}}$

Medium

Example 1:

Find the values of *x* and *y* in the expression $3^{x+1} \times 7^{2y-1} = 189$.

Solution:

It is given that $3^{x+1} \times 7^{2y-1} = 189$

$\Rightarrow 3^{x+1} \times 7^{2y-1} = 3 \times 3 \times 3 \times 7$

$\Rightarrow 3^{x+1} \times 7^{2y-1} = 3^3 \times 7^1$

On equating the exponents of 3 and 7 on both sides of the above equation, we get:

$x + 1 = 3$ and $2y - 1 = 1$

$\Rightarrow x = 3 - 1 = 2$ and $2y = 1 + 1 = 2$

$\Rightarrow x = 2$ and $y = 1$

Thus, the values of *x* and *y* are 2 and 1 respectively.

Hard

Example 1:

Simplify the expression $\left(\frac{243}{32}\right)^{\frac{-4}{5}} \times \left[\left(\frac{625}{81}\right)^{\frac{-3}{4}} \div \left(\frac{25}{4}\right)^{\frac{-3}{2}}\right]$.

Solution:

$$\left(\frac{243}{32}\right)^{\frac{-4}{5}} \times \left[\left(\frac{625}{81}\right)^{\frac{-3}{4}} \div \left(\frac{25}{4}\right)^{\frac{-3}{2}}\right]$$

$$= \left[\left(\frac{3}{2}\right)^{5}\right]^{\frac{-4}{5}} \times \left[\left\{\left(\frac{5}{3}\right)^{4}\right\}^{\frac{-3}{4}} \div \left\{\left(\frac{5}{2}\right)^{2}\right\}^{\frac{-3}{2}}\right]$$

$$= \left(\frac{3}{2}\right)^{5\times\frac{-4}{5}} \times \left\{\left(\frac{5}{3}\right)^{4\times\frac{-3}{4}} \div \left(\frac{5}{2}\right)^{2\times\frac{-3}{2}}\right\} \qquad \left[\because \left(a^{m}\right)^{n} = a^{mn}\right]$$

$$= \left(\frac{3}{2}\right)^{-4} \times \left\{\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right\}$$

$$= \left(\frac{2}{3}\right)^{4} \times \left\{\left(\frac{3}{5}\right)^{3} \div \left(\frac{2}{5}\right)^{3}\right\} \qquad \left[\because \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^{m}\right]$$

$$= \left(\frac{2}{3}\right)^{4} \times \left(\frac{3}{5} \div \frac{2}{5}\right)^{3} \qquad \left[\because a^{m} \div b^{m} = \left(a \div b\right)^{m}\right]$$

$$= \left(\frac{2}{3}\right)^{4} \times \left(\frac{3}{5} \times \frac{5}{2}\right)^{3}$$

$$= \left(\frac{2}{3}\right)^{4} \times \left(\frac{3}{2}\right)^{3}$$

$$= \left(\frac{2}{3}\right)^{4} \times \left(\frac{2}{3}\right)^{-3} \qquad \left[\because \left(\frac{b}{a}\right)^{m} = \left(\frac{a}{b}\right)^{-m}\right]$$

$$= \left(\frac{2}{3}\right)^{4-3} \qquad \left[\because a^{m} \times a^{n} = a^{m+n}\right]$$

$$= \frac{2}{3}$$

Example 2:

Prove that $\frac{x^{a(b-c)}}{x^{b(a-c)}} \div \left(\frac{x^{b}}{x^{a}}\right)^{c} = 1$.

Solution:

$$\frac{x^{a(b-c)}}{x^{b(a-c)}} \div \left(\frac{x^b}{x^a}\right)^c$$

$$= \frac{x^{(ab-ac)}}{x^{(ab-bc)}} \div \left(x^{b-a}\right)^c \qquad \left[\because \left(a^m\right)^n = a^{mn} \text{ and } \frac{a^m}{a^n} = a^{m-n}\right]$$

$$= x^{(ab-ac)-(ab-bc)} \times \frac{1}{x^{(b-a)c}} \qquad \left[\because \frac{a^m}{a^n} = a^{m-n}\right]$$

$$= x^{ab-ac-ab+bc} \times \frac{1}{x^{bc-ac}} \qquad \left[\because \left(a^m\right)^n = a^{mn}\right]$$

$$= x^{-ac+bc} \times x^{ac-bc} \qquad \left[\because \frac{1}{a^m} = a^{-m}\right]$$

$$= x^{-ac+bc+ac-bc} \qquad \left[\because a^m \times a^n = a^{m+n}\right]$$

$$= x^0$$

$$= 1$$

Concept and Laws of Logarithms

Let us suppose we are given 3 numbers: 2, 3 and 9.

Now, we know that $3^2 = 9$

Also, $\sqrt{9} = 3$

The above two expressions are formed by combining 2 and 3, and 2 and 9 respectively to get the third number.

Is there an expression wherein we can combine 3 and 9 to get 2?

3 and 9 can be combined to get 2 as:

$$\log_3 9 = 2$$

Here, 'log' is the abbreviated form of a concept called 'Logarithms'.

The expression $\log_3 9 = 2$ can be read as 'logarithm of 9 to the base 3 is equal to 2'.

In general, if *a* is any positive real number (except 1), *n* is any rational number such that $a^n = b$, then *n* is called the logarithm of *b* to the base *a*, and is written as $\log_a b$.

Thus, $a^n = b$ if and only if $\log_a b = n$.

$a^n = b$ is called the exponential form and $\log_a b = n$ is called the logarithmic form.

The following are the properties of logarithms.

1. Since *a* is any positive real number (except 1), a^n is always a positive real number for every rational number *n*, i.e., *b* is always a positive real number.

Thus, logarithms are only defined for positive real numbers.

2. Since $a^0 = 1 \Rightarrow \log_a 1 = 0$

$a^1 = a \Rightarrow \log_a a = 1$

Thus, $\log_a 1 = 0$ and $\log_a a = 1$

where, *a* is any positive real number except 1

3. If $\log_a x = \log_a y = n(\text{say})$

Then, $x = a^n$ and $y = a^n$

$\Rightarrow x = y$

Thus, $\log_a x = \log_a y$

$\Rightarrow x = y$

4. Logarithms to the base 10 are called common logarithms.

5. If no base is given, the base is always taken as 10.

For example, $\log 5 = \log_{10} 5$

Let us consider the following example.

Convert the following into logarithmic form.

(i) $5^3 = 125$

(ii) $16^{\frac{1}{2}} = 4$

$5^3 = 125 \Rightarrow \log_5 125 = 3$

$16^{\frac{1}{2}} = 4 \Rightarrow \log_{16} 4 = \frac{1}{2}$

There are three standard laws of logarithms.

(i) Product Law

$\log_a mn = \log_a m + \log_a n$

In general, $\log_a(mnp...) = \log_a m + \log_a n + \log_a p + ...$

(ii) Quotient Law

$\log_a \frac{m}{n} = \log_a m - \log_a n$

(iii) Power Law

$\log_a m^n = n \log_a m$

On the basis of the above laws, we have

For a and b two positive numbers, $\log_b a = \frac{1}{\log_a b}$.

$\Rightarrow \log_b a \times \log_a b = 1$

Also, we know that, log of a number at the same base is 1 i.e $\log_a a = 1$.

$\Rightarrow x \log_a a = x$

$\Rightarrow \log_a a^x = x$

Example 1:

Solve for *x*.

(i) $\log_7 343 = 5x - 4$

(ii) $\log_x 216 = 3$

Solution:

(i) $\log_7 343 = 5x - 4$

$$\Rightarrow 7^{(5x-4)} = 343$$
$$\Rightarrow 7^{(5x-4)} = 7^3$$
$$\Rightarrow 5x - 4 = 3$$
$$\Rightarrow 5x = 7$$
$$\Rightarrow x = \frac{7}{5}$$

(ii) $\log_x 216 = 3$

$$\Rightarrow x^3 = 216$$
$$\Rightarrow x^3 = 6^3$$
$$\Rightarrow x = 6$$

Example 2:

If $5\log\sqrt{n} - 3\log m + 1 = \log x,$ what is *x*?

Solution:

$$5\log\sqrt{n} - 3\log m + 1$$
$$= \log\left(\sqrt{n}\right)^5 - \log m^3 + \log 10$$
$$= \log\frac{\left(\sqrt{n}\right)^5 \times 10}{m^3} \qquad \begin{bmatrix} \log a^n = n\log a \\ \log a - \log b = \log\left(\frac{a}{b}\right) \\ \log a + \log b = \log ab \end{bmatrix}$$

Now, $\log x = \log\left(\frac{\sqrt{n}^5 \times 10}{m^3}\right)$

$$\Rightarrow x = \frac{10n^2\sqrt{n}}{m^3}$$

Example: 3

Solve for *x*.

(i) $\log_{13} 7 \times \log_7 13 = x$

(ii) $\log_5 5^9 = x$

Solution:

(i)
We know that,
For *a* and *b* two positive numbers, $\log_b a = \frac{1}{\log_a b}$.
$$\Rightarrow \log_b a \times \log_a b = 1$$

Therefore,

$$\log_{13} 7 \times \log_7 13 = x = 1$$
$$\Rightarrow x = 1$$

(ii)
We know that, log of a number at the same base is 1 i.e $\log_a a = 1$.
$$\Rightarrow x \log_a a = x$$
$$\Rightarrow \log_a a^x = x$$
Therefore,

$$\log_5 5^9 = x = 9$$
$$\Rightarrow x = 9$$