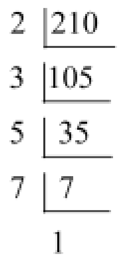
Factorisation

Factorisation of Algebraic Expressions Using Method of Common Factors

You know about the prime factorization of numbers. Let us revise the method of prime factorization by taking the example of the number 210.



We can write 210 as a product of 2, 3, 5, and 7.

Hence, 210 = 2 × 3 × 5 × 7

Here, 2, 3, 5, and 7 are the prime factors of 210. In the same way, we can factorize any expression, i.e., we can write any expression as a product of its factors.

For example, $2xyz = 2 \times x \times y \times z$

Here, 2, x, y, and z are the factors of $2xyz$, and we cannot further reduce them. Thus, we say that 2, x, y, and z are the irreducible factors of $2xyz$.

The process of writing a given algebraic expression as a product of two or more expressions is called factorization. Each of the expressions which form the product is called a factor of the given expression.

Let us discuss some more examples based on the above concept.

Example 1:

Find the common factors of the terms $6pq$, $8p^2$, and $4pq^2$.

Solution:

Write the factors of each term.

$6pq = 2 \times 3 \times p \times q$

$8p^2 = 2 \times 2 \times 2 \times p \times p$

$4pq^2 = 2 \times 2 \times p \times q \times q$

Here, 2 and p are the common factors of the given terms $6pq$, $8p^2$, and $4pq^2$.

Example 2:

Factorize $6x^2 - 18x$.

Solution:

Write the factors of each terms.

$6x^2 = 2 \times 3 \times x \times x$

$18x = 2 \times 3 \times 3 \times x$

HCF of $6x^2$ and $18x = 2 \times 3 \times x = 6x$

$\therefore 6x^2 - 18x = 6x(x - 3)$

Example 3:

Factorize the following expressions:

(i) $7x^2 + 14x$

(ii) $4a^2bcx^2y^3z^4 - 5ab^2cx^3y^4z^2 + 7abc^2x^4y^2z^3$

(iii) $-4a^2 + 3p^3 - 5b$

Solution:

(i) Write the factors of each term.

$7x^2 = 7 \times x \times x$

$14x = 2 \times 7 \times x$

Here, 7 and x are the common factors.

$7x^2 + 14x = 7 \times x \times x + 2 \times 7 \times x$

$= 7 \times x(x + 2)$

$= 7x(x + 2)$

(ii) Write the factors of each term.

$$4a^2bcx^2y^3z^4 = 2\times 2\times \underline{a}\times a\times \underline{b}\times \underline{c}\times \underline{x}\times \underline{x}\times \underline{y}\times \underline{y}\times y\times \underline{z}\times \underline{z}\times z\times z$$

$$-5ab^2cx^3y^4z^2 = -5\times \underline{a}\times \underline{b}\times b\times \underline{c}\times \underline{x}\times \underline{x}\times x\times \underline{y}\times \underline{y}\times y\times y\times \underline{z}\times \underline{z}$$

$$7abc^2x^4y^2z^3 = 7\times \underline{a}\times \underline{b}\times \underline{c}\times c\times \underline{x}\times \underline{x}\times x\times x\times \underline{y}\times \underline{y}\times \underline{z}\times \underline{z}\times z$$

Here, a, b, c, x, x, y, y, z, and z are the common factors of given terms.

Hence, $4a^2bcx^2y^3z^4 - 5ab^2cx^3y^4z^2 + 7abc^2x^4y^2z^3$

$$= a\times b\times c\times x^2\times y^2\times z^2 \begin{Bmatrix} 2\times 2\times a\times y\times z\times z - 5\times b\times x\times y\times y \\ +7\times c\times x\times x\times z \end{Bmatrix}$$

$$= abcx^2y^2z^2\left(4ayz^2 - 5bxy^2 + 7cx^2z\right)$$

(iii) Write the factors of each term.

$-4a^2 = -2\times 2\times a\times a$

$3p^3 = 3\times p\times p\times p$

$-5b = -5\times b$

There is no common factor of these terms other than 1.

Factorisation of Algebraic Expressions Using Method of Regrouping Terms

Can you factorize the algebraic expression $4ab + a + 4b + 1$?

All four terms in the expression do not have any common factor except 1. Thus, we cannot factorize the expression by taking the common factors of each term.

Let us look at the following example based on the above concept.

Example 1:

Factorize the following expressions:

(i) $2x + ax - 2y - ay$

(ii) $2a + 3b - 2 - 3ab$

Solution:

(i) The given expression is $2x + ax - 2y - ay$.

We can factorize this expression as

$2x + ax - 2y - ay = (2x - 2y) + (ax - ay)$

$= 2\ (x - y) + a\ (x - y)$

$= (x - y)\ (2 + a)$

$= (x - y)\ (2 + a)$

(ii) The given expression is $2a + 3b - 2 - 3ab$.

We can factorize this expression as

$2a + 3b - 2 - 3ab = (2a - 2) + (-\ 3ab + 3b)$

$= 2\ (a - 1) + (-3b)\ (a - 1)$

$= 2\ (a - 1) - 3b\ (a - 1)$

$= (a - 1)\ (2 - 3b)$

Factorisation of Quadratic Polynomials Using Factor Theorem and Splitting Middle Term

Factorisation of Quadratic Polynomials

We know that 7 × 6 = 42. Here, 7 and 6 are factors of 42. Now, consider the linear polynomials $x - 2$ and $x + 1$. On multiplying the two, we get: $x\ (x + 1) - 2\ (x + 1)$ $= x^2 + x - 2x - 2 = x^2 - x - 2$, which is a quadratic polynomial. So, $x - 2$ and $x + 1$ are factors of the quadratic polynomial $x^2 - x - 2$. A quadratic polynomial can have a maximum of two factors.

In the above example, we found the quadratic polynomial from its two factors. We can also find the factors from the quadratic polynomial. This process of decomposing a polynomial into a product of its factors (which when multiplied give the original expression) is called **factorisation**.

There are two ways of finding the factors of quadratic polynomials viz., by applying the factor theorem and by splitting the middle term. We will discuss these methods of factorisation in this lesson and also solve some examples based on them.

Factorisation of Quadratic Polynomials Using the Factor Theorem

The factor theorem states that: **For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(a) = 0$, then $x - a$ will be a factor of $p(x)$.**

Consider the quadratic polynomial, $p(x) = x^2 - 5x + 6$. To find its factors, we need to ascertain the value of x for which the value of the polynomial comes out to be zero. For this, we first determine the factors of the constant term in the polynomial, and then check the value of the polynomial at these points.

In the given polynomial, the constant term is 6 and its factors are ±1, ±2, ±3 and ±6.

Let us now check the value of the polynomial for each of these factors of 6.

$p(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Hence, $x - 3$ is also a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum two factors which are already obtained as: $(x - 2)$ and $(x - 3)$.

Thus, the given polynomial = $p(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$

Solved Examples

Easy

Example 1:

Factorise $x^2 - 7x + 10$ using the factor theorem.

Solution:

Let $p(x) = x^2 - 7x + 10$

The constant term is 10 and its factors are ±1, ±2, ±5 and ±10.

Let us check the value of the polynomial for each of these factors of 10.

$p(1) = 1^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(5) = 5^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$

Hence, $x - 5$ is a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.

Thus, we can write the given polynomial as:

$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$

Hard

Example 1:

Factorise $x^4y^2 - 5x^2y^2 + 6y^2$.

Solution:

$x^4y^2 - 5x^2y^2 + 6y^2 = y^2(x^4 - 5x^2 + 6)$

Let $x^2 = a$

$\Rightarrow (x^2)^2 = a^2$

$\Rightarrow x^4 = a^2$

$\therefore x^4y^2 - 5x^2y^2 + 6y^2 = y^2(a^2 - 5a + 6)$

$= y^2 \times f(a)$, where $f(a) = a^2 - 5a + 6$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term ‘6’ are ±1, ±2, ±3 and ±6.

$f(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Thus, $a - 2$ is a factor of $f(a)$.

$f(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Thus, $a - 3$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a - 2$ and $a - 3$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 5a + 6 = (a - 2)(a - 3)$

Hence, $x^4y^2 - 5x^2y^2 + 6y^2 = y^2(a - 2)(a - 3)$

$= y^2(x^2 - 2)(x^2 - 3)$

Example 2:

Factorise $4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2(y^2 - 2) - 20$.

Solution:

$4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2(y^2 - 2) - 20$

$= 4xy^2 + 4x^2 - 4x + 12 + (y^2)^2 - 2y^2 - 20$

$= (2x)^2 + (y^2)^2 + 2 \times 2x \times y^2 - 4x - 2y^2 + 12 - 20$

$= (2x + y^2)^2 - 2(2x + y^2) - 8$

$= a^2 - 2a - 8$

$= f(a)$, where $a = 2x + y^2$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term '8' are ±1, ±2, ±4 and ±8.

$f(1) = 1^2 - 2 \times 1 - 8 = 1 - 2 - 8 = -9 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(-1) = (-1)^2 - 2 \times (-1) - 8 = 1 + 2 - 8 = -5 \neq 0$

Thus, $a + 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 2 \times 2 - 8 = 4 - 4 - 8 = -8 \neq 0$

Thus, $a - 2$ is not a factor of $f(a)$.

$f(-2) = (-2)^2 - 2 \times (-2) - 8 = 4 + 4 - 8 = 0$

Thus, $a + 2$ is a factor of $f(a)$.

$f(4) = 4^2 - 2 \times 4 - 8 = 16 - 8 - 8 = 0$

Thus, $a - 4$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a + 2$ and $a - 4$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 2a - 8 = (a + 2)(a - 4)$

Hence, $4x\left(y^2 + x - 1 + \frac{3}{x}\right) + y^2(y^2 - 2) - 20 = (2x + y^2 + 2)(2x + y^2 - 4)$

Factorising Quadratic Polynomials by Splitting the Middle Terms

Example Based on the Method of Splitting the Middle Term

Solved Examples

Easy

Example 1:

Factorise $12x^2 - \sqrt{2}\,x - 12$ by splitting the middle term.

Solution:

The given polynomial is $12x^2 - \sqrt{2}\,x - 12$.

Here, $ac = 12 \times (-12) = -144$. The middle term is $-\sqrt{2}$.

Therefore, we have to split $-\sqrt{2}$ into two numbers such that their product is −144 and their sum is $-\sqrt{2}$.

These numbers are $-9\sqrt{2}$ and $8\sqrt{2}$ ($\because -9\sqrt{2} + 8\sqrt{2} = -\sqrt{2}$ and $-9\sqrt{2} \times 8\sqrt{2} = -144$).

Thus, we have:

$12x^2 - \sqrt{2}x - 12 = 12x^2 - 9\sqrt{2}x + 8\sqrt{2}x - 12$

$= 3\sqrt{2}x(2\sqrt{2}x - 3) + 4(2\sqrt{2}x - 3)$

$= (2\sqrt{2}x - 3)(3\sqrt{2}x + 4)$

Example 2:

Factorise $2x^2 - 11x + 15$ by splitting the middle term.

Solution:

The given polynomial is $2x^2 - 11x + 15$.

Here, $ac = 2 \times 15 = 30$. The middle term is −11. Therefore, we have to split −11 into two numbers such that their product is 30 and their sum is −11. These numbers are −5 and −6 [$\because (-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].

Thus, we have:

$2x^2 - 11x + 15 = 2x^2 - 5x - 6x + 15$

$= x(2x - 5) - 3(2x - 5)$

$= (2x - 5)(x - 3)$

Medium

Example 1:

Factorise $(3y - 1)^2 - 6y + 2$.

Solution:

$(3y - 1)^2 - 6y + 2 = 9y^2 + 1 - 6y - 6y + 2$

$= 9y^2 - 12y + 3$

$= 3\ (3y^2 - 4y + 1)$

Here, $ac = 1 \times 3 = 3$. The middle term is −4. Therefore, we have to split −4 into two numbers such that their product is 3 and their sum is −4. These numbers are −1 and −3 [$\because (-3) + (-1) = -4$ and $(-3) \times (-1) = 3$].

Thus, we have:

$3\ (3y^2 - 4y + 1) = 3\ (3y^2 - 3y - y + 1)$

$= 3\ [3y\ (y - 1) - 1\ (y - 1)]$

$= 3\ (y - 1)\ (3y - 1)$

Example 2:

Find the dimensions of a rectangle whose area is given by the polynomial $20p^2 + 69p + 54$.

Solution:

We know that area of a rectangle = Length × Breadth

Area of the rectangle is given by the polynomial $20p^2 + 69p + 54$. So, its factors will be the required dimensions of the rectangle.

In the given polynomial, $ac = 20 \times 54 = 1080$. The middle term is 69. Therefore, we have to split 69 into two numbers such that their product is 1080 and their sum is 69. These numbers are 45 and 24 ($\because 45 + 24 = 69$ and $45 \times 24 = 1080$).

Thus, we have:

$20p^2 + 69p + 54 = 20p^2 + 45p + 24p + 54$

$= 5p\ (4p + 9) + 6\ (4p + 9)$

$= (4p + 9)\ (5p + 6)$

Hence, the dimensions of the rectangle are $5p + 6$ and $4p + 9$.

Hard

Example 1:

Factorise $2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$.

Solution:

$$2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$$
$$=2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}\right)+9$$
$$=2\left(3x+\frac{4}{5x}\right)^2+18\left(3x+\frac{4}{5x}\right)+\left(3x+\frac{4}{5x}\right)+9$$
$$=2\left(3x+\frac{4}{5x}\right)\left[\left(3x+\frac{4}{5x}\right)+9\right]+1\left[\left(3x+\frac{4}{5x}\right)+9\right]$$
$$=\left[\left(3x+\frac{4}{5x}\right)+9\right]\left[2\left(3x+\frac{4}{5x}\right)+1\right]$$
$$=\left(3x+\frac{4}{5x}+9\right)\left(6x+\frac{8}{5x}+1\right)$$

Using Identity for "Difference of Two Squares"

Suppose we need to find the product of the numbers 79 and 81. Instead of multiplying these two numbers, we can use the identity $(a + b)(a - b)$. This identity is very important and is applicable in various situations.

Let us first understand this identity.

$(a + b)(a - b) = a(a - b) + b(a - b)$ (By distributive property)

$= a^2 - ab + ba - b^2$

$= a^2 - ab + ab - b^2$ $(ab = ba)$

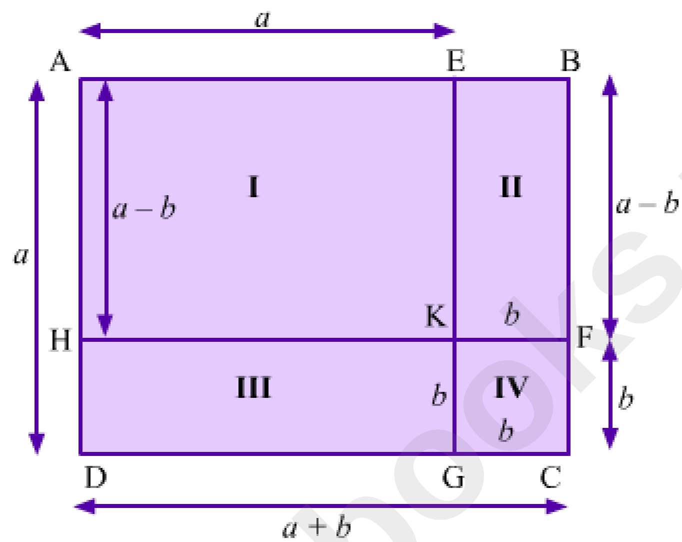
$= a^2 - b^2$

∴ $(a + b)(a - b) = a^2 - b^2$

Deriving the identity geometrically:

This identity can be derived from geometric construction as well.

For this, let us consider a square AEGD whose each side measures *a* units.



It can be seen that, a segment BC is drawn outside the square AEGD such that BC is parallel to EG and at a distance of *b* unit from G.

Another segment HF is drawn inside the square AEGD such that HF is parallel to GD and at a distance of *b* unit from G.

From the figure, it can be observed that

Area of rectangle ABFH = Area of rectangle ABCD – (Area of rectangle HKGD + Area of square KFCG)

$\Rightarrow (a + b)(a - b) = a(a + b) - [(a \times b) + (b \times b)]$

$\Rightarrow (a + b)(a - b) = a^2 + ab - ab - b^2$

$\Rightarrow \mathbf{(a + b)(a - b) = a^2 - b^2}$

Now, let us solve some examples in which the above identity can be applied.

Example 1:

Simplify the following expressions.

(a) $(x + 3)(x - 3)$

(b) $(11 + y)(11 - y)$

Solution:

(a) $(x + 3)(x - 3)$

This expression is of the form $(a + b)(a - b)$.

Hence, we can use the identity $(a + b)(a - b) = a^2 - b^2$.

$(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$

(b) $(11 + y)(11 - y)$

This expression is of the form $(a + b)(a - b)$.

Hence, we can use the identity $(a + b)(a - b) = a^2 - b^2$.

$(11 + y)(11 - y) = 11^2 - y^2 = 121 - y^2$

Example 2:

Simplify the following expressions.

(a) $\left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)$

(b) $\left(x^2-y^3\right)\left(x^2+y^3\right)+\left(y^3-z^4\right)\left(y^3+z^4\right)+\left(z^4-x^2\right)\left(z^4+x^2\right)$

Solution:

(a) The given expression is $\left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)$.

Using the identity $(a + b)(a - b) = a^2 - b^2$, we get

$$\therefore \left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)=\left(\frac{3}{7}l\right)^2-\left(\frac{4}{5}m\right)^2$$

$$=\frac{9}{49}l^2-\frac{16}{25}m^2$$

(b) $\left(x^2-y^3\right)\left(x^2+y^3\right)+\left(y^3-z^4\right)\left(y^3+z^4\right)+\left(z^4-x^2\right)\left(z^4+x^2\right)$

Using the identity (a + b) (a − b) **=** $a^2 - b^2$, we get

$$\left\{\left(x^2\right)^2-\left(y^3\right)^2\right\}+\left\{\left(y^3\right)^2-\left(z^4\right)^2\right\}+\left\{\left(z^4\right)^2-\left(x^2\right)^2\right\}$$
$$=x^4-y^6+y^6-z^8+z^8-x^4$$
$$=\left(x^4-x^4\right)+\left(-y^6+y^6\right)+\left(-z^8+z^8\right)$$
$$=0$$

Example 3:

Find the values of the following expressions using suitable identities.

(a) 195 × 205

(b) $(993)^2 - (7)^2$

(c) 24.5 × 25.5

Solution:

(a) 195 = 200 − 5
and, 205 = 200 + 5

∴ 195 × 205 = (200 − 5) × (200 + 5)

= $(200)^2 - (5)^2$ [∵∵ (a + b) (a − b) = $a^2 - b^2$]

= 40000 − 25

= 39975

(b) $(993)^2 - (7)^2$
= (993 + 7) (993 − 7) [∵ (a + b) (a − b) = $a^2 - b^2$]

= (1000) (986)

= 986000

(c) 24.5 × 25.5

= (25 − 0.5) (25 + 0.5)

= $(25)^2 - (0.5)^2$ [$\because (a + b)(a - b) = a^2 - b^2$]

= $625 - 0.25$

= 624.75

Factorisation of Algebraic Expressions Using Identities for the Sum and Difference of Two Cubes

Let us start with a simple question.

What is the value of $(203)^3$?

Yes, its value is 8365427.

For sure, you would have used the identity $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

We can write $(203)^3$ as $(200 + 3)^3$ and then using the identity for $(a + b)^3$, we can find its value as this is an easier method as compared to multiplication.

However, we have many more applications of this identity. We can also factorise algebraic expressions using this identity.

For this, we have to rewrite the identity as follows:

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$
$$\Rightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$\Rightarrow a^3 + b^3 = (a+b)\left[(a+b)^2 - 3ab\right]$$
$$\Rightarrow a^3 + b^3 = (a+b)\left[a^2 + b^2 + 2ab - 3ab\right]$$
$$\Rightarrow \boxed{a^3 + b^3 = (a+b)(a^2 + b^2 - ab)} \quad \ldots(1)$$

This form of the identity is used to factorise the expressions of the form $a^3 + b^3$.

In the same way, we can write the identity for $a^3 - b^3$ as follows:

$$\boxed{a^3 - b^3 = (a-b)(a^2 + b^2 + ab)} \quad \ldots(2)$$

To understand how to use these identities to factorise expressions, let us see an example.

Let us factorise the expression $x^6 - 729y^6$.

$x^6 - 729y^6$

$= (x^3)^2 - (27y^3)^2$

$= (x^3 + 27y^3)(x^3 - 27y^3)$[Using $a^2 - b^2 = (a + b)(a - b)$]

$= [(x)^3 + (3y)^3][(x)^3 - (3y)^3]$

Using identities (1) and (2), we obtain

$\Rightarrow (x + 3y)(x^2 + 9y^2 - 3xy)(x - 3y)(x^2 + 9y^2 + 3xy)$

This is the factorised form of the given expression.

To understand this method more clearly, let us solve some more examples.

Example 1:

Factorise the expression: $125a^6 - 343$

Solution:

$125a^6 - 343 = (5a^2)^3 - (7)^3$

$= (5a^2 - 7)(25a^4 + 49 + 35a^2)$ [Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$]

This is the factorised form of the given expression.

Example 2:

Factorise the expression: $x^3 + 64y^3 + 12x^2y + 48xy^2 - (4x + y)^3$

Solution:

$x^3 + 64y^3 + 12x^2y + 48xy^2 - (4x + y)^3$

$= x^3 + (4y)^3 + 3(x)(4y)(x + 4y) - (4x + y)^3$

$= (x + 4y)^3 - (4x + y)^3$ [Using $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$]

$= (x + 4y - 4x - y)[(x + 4y)^2 + (4x + y)^2 + (x + 4y)(4x + y)]$

[Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$]

$= (-3x + 3y)\ [x^2 + 16y^2 + 8xy + 16x^2 + y^2 + 8xy + (4x^2 + 17xy + 4y^2)$

$= 3\ (y - x)\ [21x^2 + 21y^2 + 33xy]$

$= 3\ (y - x) \times 3\ (7x^2 + 7y^2 + 11xy)$

$= 9\ (y - x)\ (7x^2 + 7y^2 + 11xy)$

This is the factorised form of the given expression.

Example 3:

Factorise the following expressions.

(1) $x^3 - \frac{1}{x^3}$

(2) $343a^3 - \frac{1}{27a^3}$

(3) $x^3 + \frac{1}{x^3}$

(4) $216m^3 + \frac{64}{m^3}$

Solution:

(1)

$x^3 - \frac{1}{x^3}$

$= (x)^3 - \left(\frac{1}{x}\right)^3$

$= \left(x - \frac{1}{x}\right)\left(x^2 + x.\frac{1}{x} + \frac{1}{x^2}\right) \quad \left[\text{Using } a^3 - b^3 = (a-b)\left(a^2 + b^2 + ab\right)\right]$

$= \left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)$

(2)

$343a^3 - \frac{1}{27a^3}$

$= (7a)^3 - \frac{1}{(3a)^3}$

$= \left(7a - \frac{1}{3a}\right)\left\{(7a)^2 + 7a.\frac{1}{3a} + \frac{1}{(3a)^2}\right\} \quad \left[\text{Using } a^3 - b^3 = (a-b)\left(a^2 + b^2 + ab\right)\right]$

$= \left(7a - \frac{1}{3a}\right)\left(49a^2 + \frac{7}{3} + \frac{1}{9a^2}\right)$

(3)

$$x^3+\frac{1}{x^3}$$

$$=(x)^3+\left(\frac{1}{x}\right)^3$$

$$=\left(x+\frac{1}{x}\right)\left(x^2-x.\frac{1}{x}+\frac{1}{x^2}\right) \qquad \left[\text{Using } a^3+b^3=(a+b)\ (a^2+b^2-ab)\right]$$

$$=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$$

(4)

$$216m^3+\frac{64}{m^3}$$

$$=(6m)^3+\left(\frac{4}{m}\right)^3$$

$$=\left(6m+\frac{4}{m}\right)\left\{(6m)^2-6m.\frac{1}{4m}+\frac{1}{(4m)^2}\right\} \qquad \left[\text{Using } a^3+b^3=(a+b)\ (a^2+b^2-ab)\right]$$

$$=\left(6m+\frac{4}{m}\right)\left(36m^2-\frac{3}{2}+\frac{1}{16m^2}\right)$$