# Expansions

Using Identities for "Square of Sum or Difference of Two Terms"

Let us try to find the square of the number 102. The square of a number, as we know, is the product of the number with itself. One way to do this is by writing the numbers one below the other, and then multiplying them as we normally do. The other way is to break the numbers and then apply distributive property. This will make our work much easier.

Let us see how.

$102^2 = 102 \times 102$

$= (100 + 2)\,(100 + 2)$

$= 100\,(100 + 2) + 2\,(100 + 2)$

$= 100 \times 100 + 100 \times 2 + 2 \times 100 + 2 \times 2$

$= 10000 + 200 + 200 + 4$

$= 10404$

Observing the similar expressions as above, we obtain the following identities.

$$(a + b)^2 = a^2 + 2ab + b^2$$

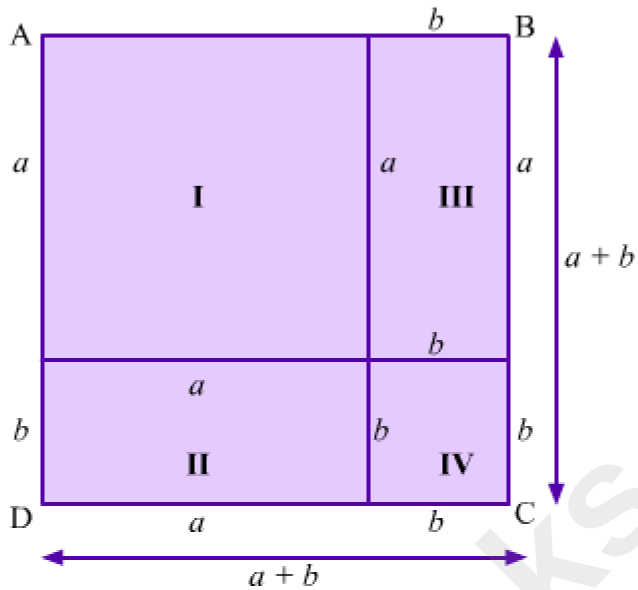
$$(a - b)^2 = a^2 - 2ab + b^2$$

**Deriving the identities geometrically:**

These identities can be derived by geometrical construction as well. Let us learn the same.

**(1) $(a + b)^2 = a^2 + 2ab + b^2$:**

Let us consider a square ABCD whose each side measures $(a + b)$ unit.






It can be seen that, we have drawn two line segments at a distance of $a$ unit from A such that one is parallel to AB and other is parallel to AD.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

∴ Area of square ABCD = $(a + b)^2$ sq. unit ...(i)

Region I is a square of side measuring $a$ unit.

∴ Area of region I = $a^2$ sq. unit ...(ii)

Each of regions II and III is a rectangle having length and breadth as $a$ unit and $b$ unit respectively.

∴ Area of region II = $ab$ sq. unit ...(iii)

And,

Area of region III = $ab$ sq. unit ...(iv)

Region IV is a square of side measuring $b$ unit.

∴ Area of region IV = $b^2$ sq. unit ...(v)

From the figure, we have

Area of square ABCD = Area of region I + Area of region II + Area of region III + Area of region IV

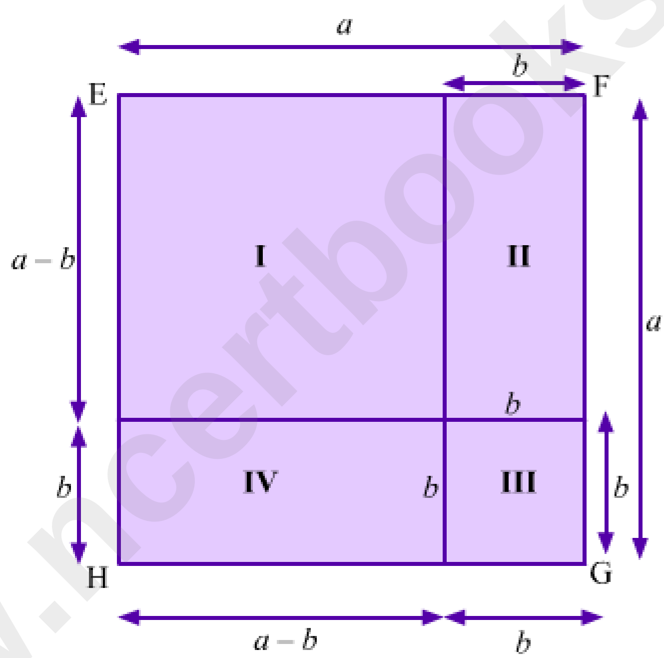
On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a + b)^2 = a^2 + ab + ab + b^2$

**$\Rightarrow (a + b)^2 = a^2 + 2ab + b^2$**

**(2) $(a - b)^2 = a^2 - 2ab + b^2$:**

Let us consider a square EFGH whose each side measures *a* unit.




It can be seen that, we have drawn two line segments at a distance of *b* unit from G such that one is parallel to GH and other is parallel to FG.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

∴ Area of square EFGH = $a^2$ sq. unit ...(i)

Region I is a square of side measuring $(a - b)$ unit.

∴ Area of region I = $(a - b)^2$ sq. unit ...(ii)

Each of regions II and IV is a rectangle having length and breadth as $(a - b)$ unit and $b$ unit respectively.

$\therefore$ Area of region II = $b(a - b)$ sq. unit ...(iii)

And,

Area of region IV = $b(a - b)$ sq. unit ...(iv)

Region III is a square of side measuring $b$ unit.

$\therefore$ Area of region III = $b^2$ sq. unit ...(v)

From the figure, we have

Area of region I = Area of square ABCD − (Area of region II + Area of region III + Area of region IV)

On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a - b)^2 = a^2 - [b(a - b) + b^2 + b(a - b)]$

$(a - b)^2 = a^2 - [ab - b^2 + b^2 + ab - b^2]$

$(a - b)^2 = a^2 - [2ab - b^2]$

$\Rightarrow (a - b)^2 = a^2 - 2ab + b^2$

The identities we have proved above are known as identity because for any value of $a$ and $b$, the LHS is always equal to the RHS. The difference between an identity and an equation is that for an equation, its LHS and RHS are equal only for some values of the variable. On the other hand, as we discussed, for an identity, the LHS equals the RHS for any value of the variable.

Many a times, these identities help in shortening our calculations. Let us discuss some examples using the above identities to understand this better.

**Example 1:**
**Simplify the following expressions using suitable identities:**
**(a) $(2m + 3n)^2$**
**(b) $(4p - 7q)^2$**

**Solution:**
(a)
On comparing the given expression $(2m + 3n)^2$ with $(a + b)^2$, we get

$a = 2m$ and $b = 3n$.
Now,
$(a + b)^2 = a^2 + 2ab + b^2$

Thus,

$$(2m+3n)^2 = (2m)^2 + 2(2m)(3n) + (3n)^2$$
$$= 4m^2 + 12mn + 9n^2$$

(b)

On comparing the given expression $(4p - 7q)^2$ with $(a - b)^2$, we get
$a = 4p$ and $b = 7q$.

Now,
$(a - b)^2 = a^2 - 2ab + b^2$

Thus,
$(4p - 7q)^2 = (4p)^2 - 2(4p)(7q) + (7q)^2$

$= 16p^2 - 56pq + 49q^2$

**Example 2:**

**Simplify the following expressions using suitable identities:**

**(a)** $(3ax + 5by)^2$

**(b)** $(0.6a^2 - 0.04b^3)^2$

**(c)** $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$

**Solution:**

**(a)** The given expression is $(3ax + 5by)^2$, which is of the form $(a+ b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$ .

$\therefore (3ax + 5by)^2 = (3ax)^2 + 2 (3ax) (5by) + (5by)^2$

$= 9a^2x^2 + 30abxy + 25b^2y^2$

**(b)** The given expression is $(0.6a^2 - 0.04b^3)^2$, which is of the form $(a - b)^2$.

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$ .

$\therefore (0.6a^2 - 0.04b^3)^2 = (0.6a^2)^2 - 2\,(0.6a^2)\,(0.04b^3) + (0.04b^3)^2$

$= 0.36a^4 - 0.048a^2b^3 + 0.0016b^6$

**(c)** The given expression is $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$ , which is of the form $(a + b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$ .

$$\therefore \left(\frac{3}{7}l + \frac{4}{5}m\right)^2 = \left(\frac{3}{7}l\right)^2 + 2\left(\frac{3}{7}l \times \frac{4}{5}m\right) + \left(\frac{4}{5}m\right)^2$$
$$= \frac{9}{49}l^2 + \frac{24}{35}lm + \frac{16}{25}m^2$$

**Example 3:**

**Find the value of $(208)^2$ using a suitable identity.**

**Solution:**

208 = 200 + 8

$\therefore (208)^2 = (200 + 8)^2$

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (208)^2 = (200 + 8)^2$

$= (200)^2 + 2\,(200)\,(8) + (8)^2$

= 40000 + 3200 + 64

= 43264

**Example 4:**

**Find the value of $(99)^2$ using a suitable identity.**

**Solution:**

99 = 100 − 1

$\therefore (99)^2 = (100 - 1)^2$

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$.

$\therefore (99)^2 = (100 - 1)^2 = (100)^2 - 2\,(100)\,(1) + (1)^2$

$= 10000 - 200 + 1$

$= 9800 + 1$

$= 9801$

**Example 5:**

**(a) If $x - \frac{1}{x} = 3$, then find the value of the expressions $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$.**

**(b) If $2y + \frac{3}{y} = 5$, then find the value of the expression $4y^2 + \frac{9}{y^2}$.**

**(c) If $3x - 5y = -1$ and $xy = 6$, then find the value of the expression $9x^2 + 25y^2$.**

**Solution:**

**(a)**It is given that $x - \frac{1}{x} = 3$.

On squaring both sides, we get

$$\left(x - \frac{1}{x}\right)^2 = 3^2$$

$$\Rightarrow (x)^2 - 2(x)\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2 = 9 \qquad \left[\text{Using the identity } (a-b)^2 = a^2 - 2ab + b^2\right]$$

$$\Rightarrow x^2 - 2 + \frac{1}{x^2} = 9$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 9 + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 11$$

Now, on squaring both sides again, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=11^2$$

$$\Rightarrow\left(x^2\right)^2+2\left(x^2\right)\left(\frac{1}{x^2}\right)+\left(\frac{1}{x^2}\right)^2=121 \qquad \left[\text{Using the identity } (a+b)^2=a^2+2ab+b^2\right]$$

$$\Rightarrow x^4+2+\frac{1}{x^4}=121$$

$$\Rightarrow x^4+\frac{1}{x^4}=119$$

Thus, the value of the expression $\left(x^2+\frac{1}{x^2}\right)$ is 11 and the value of the expression $\left(x^4+\frac{1}{x^4}\right)$ is 119.

**(b)** It is given that $2y+\frac{3}{y}=5$.

On squaring both sides, we get

$$\left(2y+\frac{3}{y}\right)^2=5^2$$

$$\Rightarrow(2y)^2+2(2y)\left(\frac{3}{y}\right)+\left(\frac{3}{y}\right)^2=25$$

$$\Rightarrow 4y^2+12+\frac{9}{y^2}=25$$

$$\Rightarrow 4y^2+\frac{9}{y^2}=13$$

Thus, the value of the expression $\left(4y^2+\frac{9}{y^2}\right)$ is 13.

**(c)** It is given that $3x-5y=-1$.

On squaring both sides, we get

$$(3x-5y)^2=(-1)^2$$
$$\Rightarrow (3x)^2-2(3x)(5y)+(5y)^2=1$$
$$\Rightarrow 9x^2-30xy+25y^2=1$$
$$\Rightarrow 9x^2-30\times 6+25y^2=1 \qquad (xy=6)$$
$$\Rightarrow 9x^2+25y^2=1+180$$
$$\Rightarrow 9x^2+25y^2=181$$

Thus, the value of the expression $(9x^2 + 25y^2)$ is 181.

**Example 6:**

**Prove that**

**(a)** $(a+b)^2+(a-b)^2=2(a^2+b^2)$

**(b)** $\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2=ab$

**Solution:**

**(a)**We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ . Thus,

LHS = $(a + b)^2 + (a - b)^2$

= $a^2 + 2ab + b^2 + a^2 - 2ab + b^2$

= $(a^2 + a^2) + (b^2 + b^2) + (2ab - 2ab)$

= $2a^2 + 2b^2$

= $2(a^2 + b^2)$ = RHS

Hence, proved.

**(b)**We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ . Thus,

$$\text{LHS}=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$$

$$= \left(\frac{a^2 + 2ab + b^2}{4}\right) - \left(\frac{a^2 - 2ab + b^2}{4}\right)$$
$$= \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{4}$$
$$= \frac{4ab}{4}$$
$$= ab = \text{RHS}$$

Hence, proved.

## Using Identities for Cube of Sum or Difference of Two Terms

**Algebraic Identities:**

**$(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$**

Consider the number '999'. Suppose we have to calculate its cube. One way to find the cube is to multiply 999 by itself three times. However, this method is tedious and, therefore, prone to error.

Here is another way to solve the problem. Let us write $999^3$ as $(1000 - 1)^3$. We have thus changed the number into the form $(x - y)^3$. Now, the expansion of $(x - y)^3$ will give the cube of 999. The required calculation will be easy since the values of $x$ and $y$ are simple numbers whose multiplication is also simple.

Thus, we see algebraic identities help make calculations simpler and less tedious. In this lesson, we will study the identities $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$. We will also solve some examples based on them.

**Understanding the Identities**

We have the two algebraic identities as follows:

- $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ OR $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$
- $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$ OR $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$

The above identities hold true for all values of the variables present in them. Let us verify this by substituting random values for the variables $x$ and $y$ in the first identity.

If $x = 2$ and $y = 3$, then:

$(2 + 3)^3 = 2^3 + 3^3 + 3 \times 2 \times 3 \times (2 + 3)$

$\Rightarrow 5^3 = 8 + 27 + 18 \times 5$

$\Rightarrow 125 = 8 + 27 + 90$

$\Rightarrow 125 = 125$

⇒ LHS = RHS

Thus, we see that the first identity holds true for random values of the variables present in it. We can prove the same for the second identity as well.

Here are some other ways in which the two identities can be represented

- $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$
- $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$ OR $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

**Proof of the Identities:**
**$(x + y)^3 = \ldots$ and $x^3 + y^3 = \ldots$**

Let us prove the identity $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$ OR $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

We can write $(x + y)^3$ as:

$(x + y)\,(x + y)^2$

$= (x + y)\,(x^2 + 2xy + y^2)$

$= x^3 + 2x^2y + 2xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + y^3 + 3x^2y + 3xy^2$

$\therefore$ **$(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$**

$\Rightarrow$ **$(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$** … (1)

Let us prove the identity $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$

We can rewrite equation 1 as:

**$x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$**

$\Rightarrow x^3 + y^3 = (x + y)\,[(x + y)^2 - 3xy]$

$\Rightarrow x^3 + y^3 = (x + y)\,(x^2 + 2xy + y^2 - 3xy)$

$\Rightarrow$ **$x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$**

## Proof of the Identities:

**$(x - y)^3 = \dots$ and $x^3 - y^3 = \dots$**

Let us prove the identity $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$ OR $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

We can write $(x - y)^3$ as:

$(x - y)\,(x - y)^2$

$= (x - y)\,(x^2 - 2xy + y^2)$

$= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3$

$= x^3 - y^3 - 3x^2y + 3xy^2$

$\therefore$ **$(x-y)^3 = x^3-y^3- 3x^2y + 3xy^2$**

$\Rightarrow$ **$(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$** … (1)

Let us prove the identity $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$ OR $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

We can rewrite equation 1 as:

**$x^3 - y^3 = (x-y)^3 + 3xy\,(x-y)$**

$\Rightarrow x^3 - y^3 = (x - y)\,[(x - y)^2 + 3xy]$

$\Rightarrow x^3 - y^3 = (x - y)\,(x^2 - 2xy + y^2 + 3xy)$

$\Rightarrow$ **$x^3 - y^3 = (x-y)\,(x^2 + xy + y^2)$**

**Example Based on the Identity $x^3 - y^3 = \dots$**

**Solved Examples**

**Easy**

**Example 1**:

Factorise the following expressions.

i)$a^3 - 125b^3 - 15a^2b + 75ab^2$

ii)$27p^3 + 125q^3$

**Solution**:

i)$a^3 - 125b^3 - 15a^2b + 75ab^2$

$= (a)^3 - (5b)^3 - 15ab\ (a - 5b)$

$= (a)^3 - (5b)^3 - 3 \times a \times 5b\ (a - 5b)$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy\ (x - y)$, where $x = a$ and $y = 5b$, we are left with $(a - 5b)^3$

$\therefore a^3 - 125b^3 - 15a^2b + 75ab^2 = (a - 5b)^3$

ii)$27p^3 + 125q^3$

$= (3p)^3 + (5q)^3$

On using the identity $x^3 + y^3 = (x + y)\ (x^2 - xy + y^2)$, where $x = 3p$ and $y = 5q$, we get:

$(3p + 5q)\ [(3p)^2 - (3p)\ (5q) + (5q)^2]$

$= (3p + 5q)\ (9p^2 - 15pq + 25q^2)$

$\therefore 27p^3 + 125q^3 = (3p + 5q)\ (9p^2 - 15pq + 25q^2)$

**Example 2**:

Evaluate the following expressions using identities.

i)$1003^3$

ii)$98^3$

**Solution**:

i)We can write $1003^3$ as:

$(1000 + 3)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy\ (x + y)$, where $x = 1000$ and $y = 3$, we get:

$(1000 + 3)^3 = 1000^3 + 3^3 + 3 \times 1000 \times 3 \times (1000 + 3)$

= 1000000000 + 27 + 9000 × (1000 + 3)

= 1000000000 + 27 + 9000000 + 27000

= 1009027027

ii)We can write $98^3$ as:

$(100 - 2)^3$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$, where $x = 100$ and $y = 2$, we get:

$(100 - 2)^3 = 100^3 - 2^3 - 3 \times 100 \times 2 \times (100 - 2)$

= 1000000 − 8 − 600 × (100 − 2)

= 1000000 − 8 − 60000 + 1200

= 941192

**Medium**

**Example 1**:

Expand the following expressions.

i) $\left(\frac{x}{a}+\frac{y}{b}\right)^3$

ii) $(2x + 5y)^3 - (2x - 5y)^3$

**Solution**:

i) $\left(\frac{x}{a}+\frac{y}{b}\right)^3$

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq\,(p + q)$, where $p = \frac{x}{a}$ and $q = \frac{y}{b}$, we get:

$$\left(\frac{x}{a}+\frac{y}{b}\right)^3 = \left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + 3\times\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)$$

$$= \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{3xy}{ab}\left(\frac{x}{a}+\frac{y}{b}\right)$$

$$= \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2}$$

ii)$(2x + 5y)^3 - (2x - 5y)^3$

We have two terms in the given expression—$(2x + 5y)^3$ and $(2x - 5y)^3$.

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq\,(p + q)$, where $p = 2x$ and $q = 5y$, we get:

$(2x + 5y)^3 = (2x)^3 + (5y)^3 + 3 \times (2x)\,(5y)\,(2x + 5y)$

$= 8x^3 + 125y^3 + 30xy\,(2x + 5y)$

$= 8x^3 + 125y^3 + 60x^2y + 150xy^2$

Similarly, on using the identity $(p - q)^3 = p^3 - q^3 - 3pq\,(p - q)$, we get

$(2x - 5y)^3 = (2x)^3 - (5y)^3 - 3 \times (2x)\,(5y)\,(2x - 5y)$

$= 8x^3 - 125y^3 - 30xy\,(2x - 5y)$

$= 8x^3 - 125y^3 - 60x^2y + 150xy^2$

So,

$(2x + 5y)^3 - (2x - 5y)^3 = 8x^3 + 125y^3 + 60x^2y + 150xy^2 - [8x^3 - 125y^3 - 60x^2y + 150xy^2]$

**=** $8x^3 + 125y^3 + 60x^2y + 150xy^2 - 8x^3 + 125y^3 + 60x^2y - 150xy^2$

**=** $250y^3 + 120x^2y$

**Alternate method**

On using the identity $p^3 - q^3 = (p - q)\,(p^2 + pq + q^2)$, where $p = (2x + 5y)$ and $q = (2x - 5y)$, we get:

$(2x + 5y)^3 - (2x - 5y)^3$

$= [(2x + 5y) - (2x - 5y)]\,[(2x + 5y)^2 + (2x + 5y)\,(2x - 5y) + (2x - 5y)^2]$

$= (2x + 5y - 2x + 5y)\ (4x^2 + 20xy + 25y^2 + 4x^2 + 10xy - 10xy - 25y^2 + 4x^2 - 20xy + 25y^2)$

$= 10y\ (12x^2 + 25y^2)$

$= 120x^2y + 250y^3$

**Example 2**:

The side of a cube is *a*. If each side of the cube is increased by b/5, then by how much does its volume increase?

**Solution**:

Let the side of the cube be *a*.

Original volume of the cube = $a \times a \times a = a^3$

After the increase, each side becomes $\left(a+\frac{b}{5}\right)$.

New volume = $\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)=\left(a+\frac{b}{5}\right)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy\ (x + y)$, where $x = a$ and $y = b/5$, we get:

$$\left(a+\frac{b}{5}\right)^3 = a^3+\left(\frac{b}{5}\right)^3+3(a)\left(\frac{b}{5}\right)\left(a+\frac{b}{5}\right)$$

$$= a^3+\frac{b^3}{125}+\frac{3ab}{5}\times\left(a+\frac{b}{5}\right)$$

$$= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$$

Now,

Increase in volume = New volume − Original volume

$$= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}-a^3$$

$$= \frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$$

Thus, the volume of the cube increases by $\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$.

**Hard**

**Example 1**:

Find the values of the following expressions.

i) $\mathbf{x^3+\frac{1}{x^3}}$ when $\mathbf{x+\frac{1}{x}=5}$

ii) $\mathbf{8y^3-\frac{27}{y^3}}$ when $\mathbf{4y^2+\frac{9}{y^2}=37}$

iii)$125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

**Solution**:

i) $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=5$

We have $x+\frac{1}{x}=5$

On cubing both sides, we get:

$$\left(x+\frac{1}{x}\right)^3=5^3$$

On using the identity $(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$, where $a = x$ and $b = \frac{1}{x}$, we get:

$$\left(x+\frac{1}{x}\right)^3=125$$

$$\Rightarrow x^3+\frac{1}{x^3}+3(x)\left(\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=125$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-3\left(x+\frac{1}{x}\right)$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-3\times 5$$

$$\Rightarrow x^3+\frac{1}{x^3}=125-15$$

$$\Rightarrow x^3+\frac{1}{x^3}=110$$

ii) $8y^3-\frac{27}{y^3}$ when $4y^2+\frac{9}{y^2}=37$

We have $4y^2+\frac{9}{y^2}=37$

$$\Rightarrow (2y)^2+\left(\frac{3}{y}\right)^2=37$$

On using the identity $a^2 + b^2 = (a - b)^2 + 2ab$, where $a = 2y$ and $b = \frac{3}{y}$, we get:

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2+2(2y)\left(\frac{3}{y}\right)=37$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2+12=37$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2=37-12$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)^2=25$$

$$\Rightarrow \left(2y-\frac{3}{y}\right)=5 \qquad \ldots(1)$$

On cubing both sides, we get:

$$\left(2y-\frac{3}{y}\right)^3=5^3$$

On using the identity $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we get:

$$(2y)^3-\left(\frac{3}{y}\right)^3-3(2y)\left(\frac{3}{y}\right)\left(2y-\frac{3}{y}\right)=125$$
$$\Rightarrow 8y^3-\frac{27}{y^3}-18\left(2y-\frac{3}{y}\right)=125$$

On using equation 1, we get:

$$8y^3-\frac{27}{y^3}-18\times 5=125$$
$$\Rightarrow 8y^3-\frac{27}{y^3}=125+90$$
$$\Rightarrow 8y^3-\frac{27}{y^3}=215$$

iii)$125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

We have $5x - 3y = 1$

On cubing both sides, we get:

$(5x - 3y)^3 = 1^3$

On using the identity $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, where $a = 5x$ and $b = 3y$, we get:

$(5x)^3 - (3y)^3 - 3(5x)(3y)(5x - 3y) = 1$

$\Rightarrow 125x^3 - 27y^3 - 45xy(5x - 3y) = 1$

On substituting the values of $xy$ and $(5x - 3y)$,we get:

$125x^3 - 27y^3 - 45 \times 6 \times 1 = 1$

$\Rightarrow 125x^3 - 27y^3 = 1 + 270$

$\Rightarrow 125x^3 - 27y^3 = 271$

**Example 2**:

If $x + y = 8$ and $x^2 + y^2 = 42$, then find the value of $x^3 + y^3$.

**Solution**:

It is given that $x + y = 8$

$\Rightarrow (x + y)^2 = 8^2$

$\Rightarrow x^2 + y^2 + 2xy = 64$

$\Rightarrow 42 + 2xy = 64$ ($\because x^2 + y^2 = 42$)

$\Rightarrow 2xy = 64 - 42$

$\Rightarrow 2xy = 22$

$\Rightarrow xy = \frac{22}{2}$

$\Rightarrow \therefore xy = 11$

Now, $(x + y)^3 = 8^3$

$\Rightarrow x^3 + y^3 + 3xy(x + y) = 512$

$\Rightarrow x^3 + y^3 + 3 \times 11 \times 8 = 512$

$\Rightarrow x^3 + y^3 + 264 = 512$

$\Rightarrow x^3 + y^3 = 512 - 264$

$\Rightarrow \therefore x^3 + y^3 = 248$

**Example 3**:

Prove that $\frac{0.77 \times 0.77 \times 0.77 + 0.23 \times 0.23 \times 0.23}{0.77 \times 0.77 - 0.77 \times 0.23 + 0.23 \times 0.23} = 1$.

**Solution**:

$$\frac{0.77\times0.77\times0.77+0.23\times0.23\times0.23}{0.77\times0.77-0.77\times0.23+0.23\times0.23}$$
$$=\frac{(0.77)^3+(0.23)^3}{(0.77)^2-0.77\times0.23+(0.23)^2}$$
$$=\frac{x^3+y^3}{x^2-xy+y^2},\text{ where } x=0.77 \text{ and } y=0.23$$
$$=\frac{(x+y)(x^2-xy+y^2)}{(x^2-xy+y^2)}$$
$$=(0.77+0.23)$$
$$=1$$

## Using Identity (x + a)(x + b)

A very important identity that we have to learn is regarding the expression $(x + a)(x + b)$.

We know how to multiply binomials. By finding the product of these binomials, we can find the identity.

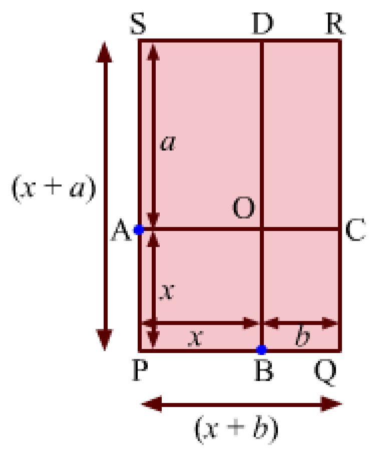
This identity can be derived by geometrical construction as well. Let us learn the same.

**Deriving the identity geometrically:**

Let us draw a rectangle PQRS of length and breadth $(x + a)$ and $(x + b)$ units respectively.

Also, let us take two points A and B on sides SP and PQ respectively, such that PA = PB = $x$ unit.

Now, let us draw two line segments AC and BD such that AC || PQ and BD || PS. AC intersects QR at point C and BD intersects RS at point D.




Now, we have

$l$(PS) = ($x$ + $a$) unit and $l$(PQ) = ($x$ + $b$) unit

∴ Area of rectangle PQRS = length × breadth

⇒ Area of rectangle PQRS = $l$(PS) × $l$(PQ)

⇒ Area of rectangle PQRS = $(x + a)(x + b)$ sq. unit ...(i)

Also,

Area of square PBOA = $x^2$ sq. unit ...(ii)

Area of rectangle BQCO = $bx$ sq. unit ...(iii)

Area of rectangle OCRD = $ab$ sq. unit ...(iv)

Area of rectangle AODS = $ax$ sq. unit ...(v)

From the figure, it can be observed that

Area of rectangle PQRS = Area of square PBOA + Area of rectangle BQCO + Area of rectangle OCRD + Area of rectangle AODS

On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(x + a)(x + b) = x^2 + bx + ab + ax$

$\Rightarrow (x + a)(x + b) = x^2 + ax + bx + ab$

**⇒ $(x + a)(x + b) = x^2 +$**
**$(a + b)x + ab$**

Now, let us solve some examples in which this identity is used.

**Example 1:**
Find the product of $(m + 3)$ and $(m – 5)$.

**Solution:**

This expression is of the form $(x + a)(x + b)$.

Using identity, $(x + a)(x + b) = x^2 + (a + b)x + ab$

$$\therefore (m+3)(m-5) = m^2 + (3-5)m + (3)(-5)$$
$$= m^2 - 2m - 15$$

**Example 2:**

**Use the appropriate identity to simplify the following expressions.**

**(a)** $(3p+5q)(3p-7z)$

**(b)** $\left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right)$

**Solution:**

**(a)**The given expression is $(3p+5q)(3p-7z)$.

This expression is of the form $(x + a)(x + b)$.

Using identity, $(x + a)(x + b) = x^2 + (a + b)x + ab$

$$\therefore (3p+5q)(3p-7z) = (3p)^2 + (5q-7z)(3p) + (5q)(-7z)$$
$$= 9p^2 + (5q)(3p) - (7z)(3p) - 35qz$$
$$= 9p^2 + 15pq - 21pz - 35qz$$

**(b)**The given expression is $\left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right)$.

This expression is of the form (*x* + *a*) (*x* + *b*).

Using identity, $(x + a)(x + b) = x^2 + (a + b)x + ab$

$$\therefore \left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right)=\left(z^2\right)^2+\left\{\left(-\frac{4}{3}\right)+\left(-\frac{3}{2}\right)\right\}z^2+\left(-\frac{4}{3}\right)\times\left(-\frac{3}{2}\right)$$
$$=z^4+\left(\frac{-8-9}{6}\right)z^2+2$$
$$=z^4-\frac{17}{6}z^2+2$$

**Example 3:**

**Find the products of the following pairs of numbers using suitable identities.**

**(a) 105 × 102**

**(b) 98 × 103**

**Solution:**

**(a)** 105 × 102 = (100 + 5) × (100 + 2)

Now, we can use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$

Here, $x = 100$, $a = 5$, $b = 2$

∴105 × 102 = (100 + 5) × (100 + 2)

$= (100)^2 + (5 + 2) \times 100 + 5 \times 2$

= 10000 + 700 + 10

= 10710

Thus, the product of the numbers 105 and 102 is 10710.

**(b)** 98 × 103 = (100 − 2) × (100 + 3)

Now, we can use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$

Here, $x = 100$, $a = -2$, $b = 3$

$\therefore 98 \times 103 = (100 - 2) \times (100 + 3)$

$= (100)^2 + (-2 + 3) \times 100 + (-2)(3)$

$= 10000 + 1 \times 100 - 6$

$= 10000 + 100 - 6$

$= 10100 - 6$

$= 10094$

Thus, the product of the numbers 98 and 103 is 10094.

## Using Identity for Square of Sum of Three Terms

**Algebraic Identity:**
$\boldsymbol{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$

When we solve an **algebraic equation**, we get the values of the variables present in it. When an algebraic equation is valid for all values of its variables, it is called an **algebraic identity**.

So, an algebraic identity is a relation that holds true for all possible values of its variables. We can use algebraic identities to expand, factorise and evaluate various algebraic expressions.

Many algebraic identities are used in mathematics. One such identity is$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. In this lesson, we will study this identity and solve some examples based on it.

**Proof of the Identity**

Let us prove the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

We can write $(x + y + z)^2$ as :

$(a + z)^2$, where $a = x + y$

$= a^2 + 2az + z^2$ [Using the identity $(x + y)^2 = x^2 + 2xy + y^2$]

$= (x + y)^2 + 2(x + y)z + z^2$ (Substituting the value of $a$)

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$ (Using the identity $(x + y)^2 = x^2 + 2xy + y^2$)

**$\therefore (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$**

The above identity holds true for all values of the variables present in it. Let us verify this by substituting random values for the variables *x*, *y* and *z*.

If $x = 2$, $y = 3$ and $z = 4$, then:

$(2 + 3 + 4)^2 = 2^2 + 3^2 + 4^2 + 2 \times 2 \times 3 + 2 \times 3 \times 4 + 2 \times 4 \times 2$

$\Rightarrow 9^2 = 4 + 9 + 16 + 12 + 24 + 16$

$\Rightarrow 81 = 81$

$\Rightarrow$ LHS = RHS

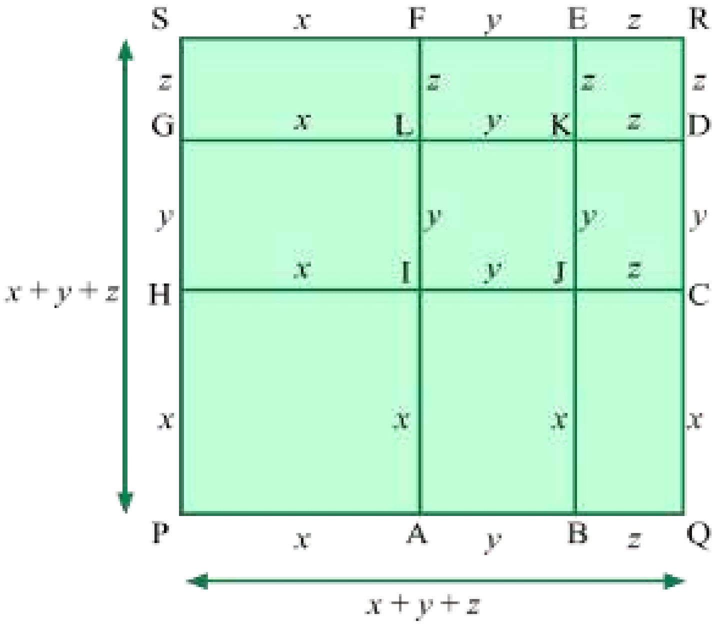
Thus, we see that the identity holds true for random values of the variables present in it.

Let us now use this identity to expand, factorise and evaluate various algebraic expressions.

**Deriving Identity Geometrically**

The identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ can also be derived with the help of geometrical construction.

The steps of construction are as follows:




(1) Draw a square PQRS of side measuring ($x + y + z$) taking any convenient values of $x$, $y$ and $z$.

(2) Mark two points A and B on side PQ such that $l$(PA) = $x$ and $l$(AB) = $y$. Thus, $l$(BQ) = $z$. Also, mark two points H and G on side PS such that $l$(PH) = $x$ and $l$(HG) = $y$. Thus, $l$(GS) = $z$.

(3) From points A and B, draw segments AF and BE parallel to side PS and intersecting RS at F and E respectively.

(4) From points H and G, draw segments HC and GD parallel to side PQ and intersecting QR at C and D respectively.

## From the figure, it can be observed that

Area of square PQRS = Sum of areas of squares PAIH, IJKL and KDRE + Sum of areas of rectangles ABJI, BQCJ, JCDK, HILG, LKEF and GLFS

$\Rightarrow (x + y + z)^2 = (x^2 + y^2 + z^2) + (xy + zx + yz + xy + yz + zx)$

$\Rightarrow \mathbf{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$

**Solved Examples**

**Easy**

**Example 1**:

Expand the following expressions.

i) $(ab - bc + ca)^2$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

**Solution**:

i) $(ab - bc + ca)^2$

On comparing the expression $(ab - bc + ca)^2$ with $(x + y + z)^2$, we get:

$x = ab$, $y = -bc$ and $z = ca$

On using the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get:

$(ab)^2 + (-bc)^2 + (ca)^2 + 2(ab)(-bc) + 2(-bc)(ca) + 2(ca)(ab)$

$= a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

$\therefore (ab - bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

On comparing the expression $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$ with $(a + b + c)^2$, we get:

$a = \frac{1}{2}x$, $b = -\frac{2}{3}y$ and $c = -\frac{3}{4}$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, we get:

$$\left(\frac{1}{2}x\right)^2 + \left(-\frac{2}{3}y\right)^2 + \left(-\frac{3}{4}\right)^2 + 2\left(\frac{1}{2}x\right)\left(-\frac{2}{3}y\right) + 2\left(-\frac{2}{3}y\right)\left(-\frac{3}{4}\right) + 2\left(-\frac{3}{4}\right)\left(\frac{1}{2}x\right)$$

$$\Rightarrow \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{9}{16} - \frac{2}{3}xy + y - \frac{3}{4}x$$

$$\therefore \left(\frac{1}{2}x-\frac{2}{3}y-\frac{3}{4}\right)^2=\frac{1}{4}x^2+\frac{4}{9}y^2+\frac{9}{16}-\frac{2}{3}xy+y-\frac{3}{4}x$$

**Example 2**:

Expand the expression $(xy + yz + zx)^2$.

**Solution**:

On comparing the expression $(xy + yz + zx)^2$ with $(a + b + c)^2$, we get:

$a = xy$, $b = yz$ and $c = zx$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,we get:

$(xy)^2 + (yz)^2 + (zx)^2 + 2(xy)(yz) + 2(yz)(zx) + 2(zx)(xy)$

$= x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

$\therefore (xy + yz + zx)^2 = x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

**Medium**

**Example 1**:

Factorize the following expressions.

i)$8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27$

ii)$8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18$

**Solution**:

i)$8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27$

**=** $8x^2 + 12y^2 + 27 - 8\sqrt{6}xy - 36y + 12\sqrt{6}x$

$= (2\sqrt{2}x)^2 + (-2\sqrt{3}y)^2 + (3\sqrt{3})^2 + 2(2\sqrt{2}x)(-2\sqrt{3}y) + 2(-2\sqrt{3}y)(3\sqrt{3}) + 2(3\sqrt{3})(2\sqrt{2}x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2\sqrt{2}x$, $b = -2\sqrt{3}y$ and $c = 3\sqrt{3}$, we are left with $(2\sqrt{2}x - 2\sqrt{3}y + 3\sqrt{3})^2$

$\therefore 8x^2 + 12y^2 - 8\sqrt{6}\,xy + 12\sqrt{6}\,x - 36y + 27 = (2\sqrt{2}\,x - 2\sqrt{3}\,y + 3\sqrt{3})^2$

ii) $8x^4 + 4\sqrt{2}\,x^3 + 25x^2 + 6\sqrt{2}\,x + 18$

$= 8x^4 + 25x^2 + 18 + 4\sqrt{2}\,x^3 + 6\sqrt{2}\,x$

$= 8x^4 + x^2 + 18 + 4\sqrt{2}\,x^3 + 6\sqrt{2}\,x + 24x^2$ ($\because 25x^2 = x^2 + 24x^2$)

$= (2\sqrt{2}\,x^2)^2 + (x)^2 + (3\sqrt{2})^2 + 2(2\sqrt{2}\,x^2)(x) + 2(x)(3\sqrt{2}) + 2(3\sqrt{2})(2\sqrt{2}\,x^2)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2\sqrt{2}\,x^2$, $b = x$ and $c = 3\sqrt{2}$, we are left with $(2\sqrt{2}\,x^2 + x + 3\sqrt{2})^2$

$\therefore 8x^4 + 4\sqrt{2}\,x^3 + 25x^2 + 6\sqrt{2}\,x + 18 = (2\sqrt{2}\,x^2 + x + 3\sqrt{2})^2$

**Example 2**:

Find the value of the expression $4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$ for $x = 3$, $y = 4$ and
$z = 5$ without substituting the values of the variables in the expression.

**Solution**:

$4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$

$= (2x)^2 + (-3y)^2 + (4z)^2 + 2(2x)(-3y) + 2(-3y)(4z) + 2(4z)(2x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2x$, $b = -3y$ and $c = 4z$, we are left with $(2x - 3y + 4z)^2$

It is given that $x = 3$, $y = 4$ and $z = 5$.

On substituting the values of $x$, $y$ and $z$, we get:

$(2 \times 3 - 3 \times 4 + 4 \times 5)^2$

$= (6 - 12 + 20)^2$

$= 14^2$

$= 196$

∴ $4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx = 196$

**Hard**

**Example 1**:

Find the value of $ab + bc + ca$, where $a + b + c = 1$ and $a^2 + b^2 + c^2 = 29$.

**Solution**:

We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow (1)^2 = 29 + 2(ab + bc + ca)$

$\Rightarrow 1 - 29 = 2(ab + bc + ca)$

$\Rightarrow -28 = 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca = -\frac{28}{2}$

$\Rightarrow \therefore ab + bc + ca = -14$

**Example 2**:

If $(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x(6 + a) + 2y(2 - a) - 8z + 2(xy + yz + xz) - 8(3 - a)$

$= (x + y + z)^2$, then find the value of $a$.

**Solution**:

$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x(6 + a) + 2y(2 - a) - 8z + 2(xy + yz + xz) - 8(3 - a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 8z + 2xy + 2yz + 2xz - 24 + 8a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 12z + 4z + 2xy + 2yz + 2xz - 24 + 12a - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2xy - 12x + 4y - 24 + 2yz - 2ay - 12z + 12a + 2xz - 2ax + 4z - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2(xy - 6x + 2y - 12) + 2(yz - ay - 6z + 6a) + 2(xz - ax + 2z - 2a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2[x(y - 6) + 2(y - 6)] + 2[y(z - a) - 6(z - a)] + 2[x(z - a) + 2(z - a)]$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2 (x + 2) (y - 6) + 2 (y - 6) (z - a) + 2 (x + 2) (z - a)$

$= [(x +2) + (y - 6) + (z - a)]^2$

$= (x + y + z - 4 - a)^2$

It is given that
$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x (6 + a) + 2y (2 - a) - 8z + 2 (xy + yz + xz) - 8 (3 - a)$

$= (x + y + z)^2$

$\Rightarrow (x + y + z - 4 - a)^2 = (x + y + z)^2$

$\Rightarrow -4 - a = 0$

$\Rightarrow \therefore a = -4$

## Conditional Identities

An identity is an equation that is true for all the values of the variables present in it.

Some well-known identities are as follows:

• $(a + b)^2 \equiv a^2 + 2ab + b^2$

• $(a - b)^2 \equiv a^2 - 2ab + b^2$

• $(a - b)^3 \equiv a^3 - b^3 - 3ab (a - b)$

We use the equivalence symbol (≡) to denote algebraic identities.

The above identities hold true for all real values of $a$ and $b$.

**If an equation is true for all values of the variables present in it only under a certain condition, then the equation is known as a conditional identity.**

Let us understand this with the help of an example.

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Here, the equation $a^3 + b^3 + c^3 = 3abc$ is true for all real values of variables $a$, $b$ and $c$ under the condition $a + b + c = 0$. This means that if $a + b + c \neq 0$, then the identity $a^3 + b^3 + c^3 = 3abc$ does not hold true.

Thus, $a^3 + b^3 + c^3 = 3abc$ is a conditional identity.

Let us prove this.

We know that:

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

When $a + b + c = 0$, the above equation reduces as follows:

$a^3 + b^3 + c^3 - 3abc = 0 \times (a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

Thus, the identity holds true under the condition $a + b + c = 0$.

Conditional identities can be very useful while solving certain algebraic problems.

Let us solve a couple of problems related to conditional identities.

**Example 1**:

**If $a + 2b + 3c = 0$, then prove that $(2b + 3c)(2b - 3c) + a(a + 4b) = 0$.**

**Solution**:

$$\begin{aligned}\text{LHS} &= (2b+3c)(2b-3c)+a(a+4b)\\ &= 4b^2-9c^2+a^2+4ab\\ &= a^2+4ab+4b^2-9c^2\\ &= (a+2b)^2-(3c)^2\\ &= (a+2b+3c)(a+2b-3c)\\ &= 0 \qquad [\because a+2b+3c=0]\\ &= \text{RHS}\end{aligned}$$

Hence proved.

**Example 2**:

**If *a* + 2*b* + *c* = 0, then prove that** $\dfrac{a^2}{2bc}+\dfrac{4b^2}{ca}+\dfrac{c^2}{2ab}=3$**.**

**Solution**:

$$\begin{aligned}\text{LHS} &= \frac{a^2}{2bc}+\frac{4b^2}{ca}+\frac{c^2}{2ab}\\ &= \frac{a^3+8b^3+c^3}{2abc}\\ &= \frac{a^3+(2b)^3+c^3}{2abc}\end{aligned}$$

We know that:

$$a^3+(2b)^3+c^3-3a(2b)c=(a+2b+c)\left\{a^2+(2b)^2+c^2-a(2b)-(2b)c-ca\right\}$$

When *a* + 2*b* + *c* = 0, we get $a^3+8b^3+c^3=3a(2b)c=6abc$

Therefore,

$$\begin{aligned}\text{LHS} &= \frac{6abc}{2abc}\\ &= 3\\ &= \text{RHS}\end{aligned}$$

Hence proved.

**Solving Problems Using the Identity (x + y + z) (x2 + y2 + z2 − xy − yz − zx)**

**Algebraic Identity:**

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Algebraic identities help us solve problems with ease and in minimum time. Say, for example, we need to find the value of $(-32^3 + 15^3 + 17^3)$. One may solve this problem by calculating the cube of each of the given numbers and then adding and subtracting the values so obtained. This method is easy in cases where we are dealing with small numbers. However, when big numbers are involved (as in the present case), this method proves to be tedious.

A simpler and less time-consuming way of solving the above problem is to use an appropriate algebraic identity. In the given expression, we find that $-32 + 15 + 17 = 0$. So, we need to state an identity under the condition $x + y + z = 0$.

In this lesson, we will focus on the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ and its expansion under the condition $x + y + z = 0$. We will also solve examples based on the same.

**Understanding the Identity**

We have the algebraic identity as follows:

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

***Or***

$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

The above identity holds true for all values of the variables present in it. Let us verify this by substituting random values for the variables $x$, $y$ and $z$.

If $x = 1$, $y = 2$ and $z = 3$, then:

$1^3 + 2^3 + 3^3 - 3 \times 1 \times 2 \times 3 = (1 + 2 + 3)(1^2 + 2^2 + 3^2 - 1 \times 2 - 2 \times 3 - 3 \times 1)$

$\Rightarrow 1 + 8 + 27 - 18 = 6(1 + 4 + 9 - 2 - 6 - 3)$

$\Rightarrow 18 = 6 \times 3$

$\Rightarrow 18 = 18$

$\Rightarrow$ LHS = RHS

Thus, we see that the identity holds true for random values of the variables present in it.

**Proof of the Identity**

Let us prove the identity

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

***Or***

$x^3 + y^3 + z^3 - 3xyz = 1/2\ (x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

We can write $x^3 + y^3 + z^3 - 3xyz$ as:

$(x^3 + y^3) + z^3 - 3xyz$

$= [(x + y)^3 - 3xy(x + y)] + z^3 - 3xyz$

$= a^3 - 3axy + z^3 - 3xyz$, where $a = x + y$

$= (a^3 + z^3) - 3axy - 3xyz$

$= (a + z)(a^2 - az + z^2) - 3xy(a + z)$

$= (a + z)(a^2 - az + z^2 - 3xy)$

$= (x + y + z)[(x + y)^2 - (x + y)z + z^2 - 3xy]$

$= (x + y + z)(x^2 + y^2 + 2xy - zx - yz + z^2 - 3xy)$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

**$\therefore x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$**

On multiplying and dividing the above expanded form by 2, we get:

$1/2 \times 2 (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= 1/2 (x + y + z)(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$

$= 1/2 (x + y + z)(x^2 + x^2 + y^2 + y^2 + z^2 + z^2 - 2xy - 2yz - 2zx)$

$= 1/2 (x + y + z)(x^2 + y^2 - 2xy + y^2 + z^2 - 2yz + z^2 + x^2 - 2zx)$

$= 1/2 (x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

**$\therefore x^3 + y^3 + z^3 - 3xyz = 1/2 (x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$**

**Example Based on the Identity**

**Solved Examples**

**Easy**

**Example 1**:

Factorize the following expressions.

i) $125x^3 + 8y^3 + 27z^3 - 90xyz$

ii) $343p^3 - 64y^3 + 8 + 168py$

**Solution**:

i) $125x^3 + 8y^3 + 27z^3 - 90xyz$

$= (5x)^3 + (2y)^3 + (3z)^3 - 3\,(5x)\,(2y)\,(3z)$

On using the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$, we get:

$(5x + 2y + 3z)\,[(5x)^2 + (2y)^2 + (3z)^2 - (5x)\,(2y) - (2y)\,(3z) - (3z)\,(5x)]$

$= (5x + 2y + 3z)\,(25x^2 + 4y^2 + 9z^2 - 10xy - 6yz - 15xz)$

$\therefore 125x^3 + 8y^3 + 27z^3 - 90xyz = (5x + 2y + 3z)\,(25x^2 + 4y^2 + 9z^2 - 10xy - 6yz - 15xz)$

ii) $343p^3 - 64y^3 + 8 + 168py$

$= (7p)^3 + (-4y)^3 + (2)^3 - 3\,(7p)\,(-4y)\,(2)$

On using the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$, we get:

$[7p + (-4y) + 2]\,[(7p)^2 + (-4y)^2 + (2)^2 - (7p)\,(-4y) - (-4y)\,(2) - (2)\,(7p)]$

$= (7p - 4y + 2)\,(49p^2 + 16y^2 + 4 + 28py + 8y - 14p)$

$\therefore 343p^3 - 64y^3 + 8 + 168py = (7p - 4y + 2)\,(49p^2 + 16y^2 + 4 + 28py + 8y - 14p)$

**Medium**

**Example 1**:

If $a + b + c = 10$ and $ab + bc + ca = 31$, then find the value of $a^3 + b^3 + c^3 - 3abc$.

**Solution**:

We know that $a^3 + b^3 + c^3 - 3abc = (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$

Or, $a^3 + b^3 + c^3 - 3abc = (a + b + c)\,[a^2 + b^2 + c^2 - (ab + bc + ca)]$ … (1)

It is given that $a + b + c = 10$ and $ab + bc + ca = 31$.

Let us find the value of $a^2 + b^2 + c^2$.

We have the identity:

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow (10)^2 = a^2 + b^2 + c^2 + 2 \times 31$

$\Rightarrow 100 = a^2 + b^2 + c^2 + 62$

$\Rightarrow a^2 + b^2 + c^2 = 100 - 62$

$\Rightarrow a^2 + b^2 + c^2 = 38$

On substituting all the values in equation 1, we obtain:

$a^3 + b^3 + c^3 - 3abc = 10 \times (38 - 31)$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 10 \times 7$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 70$

**Example 2:**

Factorise the expression $\frac{x^3}{8} - \frac{27}{x^3} - \frac{11}{2}$.

**Solution:**

We have

$$\frac{x^3}{8} - \frac{27}{x^3} - \frac{11}{2}$$

$$= \frac{x^3}{8} - \frac{27}{x^3} - 1 - \frac{9}{2}$$

$$= \left(\frac{x}{2}\right)^3 + \left(-\frac{3}{x}\right)^3 + (-1)^3 - 3\left(\frac{x}{2}\right)\left(-\frac{3}{x}\right)(-1)$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\left(\frac{x}{2}\right)^2 + \left(-\frac{3}{x}\right)^2 + (-1)^2 - \left(\frac{x}{2}\right)\left(-\frac{3}{x}\right) - \left(-\frac{3}{x}\right)(-1) - (-1)\left(\frac{x}{2}\right)\right\}$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\frac{x^2}{4} + \frac{9}{x^2} + 1 + \frac{3}{2} - \frac{3}{x} + \frac{x}{2}\right\}$$

$$= \left(\frac{x}{2} - \frac{3}{x} - 1\right)\left\{\frac{x^2}{4} + \frac{x}{2} + \frac{5}{2} - \frac{3}{x} + \frac{9}{x^2}\right\}$$

**Case I of the Identity**

A special case for the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)$
$(x^2 + y^2 + z^2 - xy - yz - zx)$ is given below.

**Case: When $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$.**

**Proof:** We have,

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On substituting $x + y + z = 0$, we obtain

$x^3 + y^3 + z^3 - 3xyz = 0 \times (x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^3 + y^3 + z^3 - 3xyz = 0$

$\Rightarrow x^3 + y^3 + z^3 = 3xyz$

Using this condition, we can factorize and find the values of many complex expressions.

**Example Based on the Case I of the Identity**

**Solved Examples**

**Easy**

**Example 1**:

Without actually calculating the cubes, find the value of each of the following expressions.

i) $(0.2)^3 - (0.5)^3 + (0.3)^3$

ii) $-12^3 + 25^3 - 13^3$

**Solution**:

i) We can write $(0.2)^3 - (0.5)^3 + (0.3)^3$ as $(0.2)^3 + (-0.5)^3 + (0.3)^3$.

Let us consider $x = 0.2$, $y = -0.5$ and $z = 0.3$.

Now, $x + y + z = 0.2 - 0.5 + 0.3 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

On substituting the values of $x$, $y$ and $z$, we obtain:

$(0.2)^3 + (-0.5)^3 + (0.3)^3 = 3 \times 0.2 \times (-0.5) \times 0.3$

$\Rightarrow (0.2)^3 - (0.5)^3 + (0.3)^3 = -0.09$

ii) We can write $-12^3 + 25^3 - 13^3$ as $(-12)^3 + 25^3 + (-13)^3$.

Let us consider $x = -12$, $y = 25$ and $z = -13$.

Now, $x + y + z = -12 + 25 - 13 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

On substituting the values of $x$, $y$ and $z$, we obtain:

$(-12)^3 + 25^3 + (-13)^3 = 3 \times (-12) \times 25 \times (-13)$

$\Rightarrow -12^3 + 25^3 - 13^3 = 11700$

**Medium**

**Example 1**:

Find the value of the expression $8x^3 + 27y^3 - 64z^3$ without directly substituting the values $x = 3$, $y = 2$ and $z = 3$.

**Solution**:

We can write $8x^3 + 27y^3 - 64z^3$ as $(2x)^3 + (3y)^3 + (-4z)^3$.

For the given values of $x$, $y$ and $z$, we get:

$(2x) + (3y) + (-4z) = 2 \times 3 + 3 \times 2 - 4 \times 3$

$\Rightarrow (2x) + (3y) + (-4z) = 6 + 6 - 12$

$\Rightarrow (2x) + (3y) + (-4z) = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we have:

$(2x)^3 + (3y)^3 + (-4z)^3 = 3\,(2x)\,(3y)\,(-4z)$

$\Rightarrow 8x^3 + 27y^3 - 64z^3 = -72xyz$

On substituting the values of $x$, $y$ and $z$, we obtain:

$8x^3 + 27y^3 - 64z^3 = -72 \times 3 \times 2 \times 3$

$\Rightarrow 8x^3 + 27y^3 - 64z^3 = -1296$

**Hard**

**Example 1**:

Show that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3} = (a+b)(b+c)(c+a)$.

**Solution**:

We will factorize the numerator and the denominator separately.

We have the numerator as $(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3$

Let us consider $x = a^2 - b^2$, $y = b^2 - c^2$ and $z = c^2 - a^2$.

Now, $x + y + z = a^2 - b^2 + b^2 - c^2 + c^2 - a^2 = 0$

We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we obtain:

$(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$

On using the identity $x^2 - y^2 = (x + y)(x - y)$, we get:

$(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a - b)(a + b)(b - c)(b + c)(c - a)(c + a)$

We have the denominator as $(a - b)^3 + (b - c)^3 + (c - a)^3$.

Let us consider $x = a - b$, $y = b - c$ and $z = c - a$.

Now, $x + y + z = a - b + b - c + c - a = 0$

Again, if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Thus, we obtain:

$(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$

On putting back the numerator and the denominator, we get:

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=\frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$$

$$\Rightarrow \frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$$

**Case II of the Identity**

One more special case of the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ is there which is explained below.

**Case: When $x + y + z \neq 0$ and $x^3 + y^3 + z^3 - 3xyz = 0$ then $x = y = z$.**

**Proof:** We have,

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On substituting $x^3 + y^3 + z^3 - 3xyz = 0$, we obtain

$0 = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^2 + y^2 + z^2 - xy - yz - zx = 0 \qquad (x + y + z \neq 0)$

$\Rightarrow \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

$\Rightarrow [(x - y)^2 + (y - z)^2 + (z - x)^2] = 0 \qquad (x + y + z \neq 0)$

Since, the sum of non negative terms such as $(x - y)^2$, $(y - z)^2$ and $(z - x)^2$ is 0, each term is 0.

$\therefore (x - y)^2 = 0$, $(y - z)^2 = 0$ and $(z - x)^2 = 0$

$\Rightarrow x - y = 0$, $y - z = 0$ and $z - x = 0$

$\Rightarrow x = y$, $y = z$ and $z = x$

$\Rightarrow x = y = z$

This condition can be very helpful to factorize and find the values of many complex expressions.

**Example Based on the Case II of the Identity**

**Example** :

If $2a = 3b = 4c = 24$ then without actually calculating the cubes of $a$, $b$ and $c$, find the value of $8a^3 + 27b^3 + 64c^3$.

**Solution**:

We have,

$2a = 3b = 4c = 24$

$\Rightarrow a = 12$, $b = 8$ and $c = 6$

Also, $2a + 3b + 4c = 72 \neq 0$

Therefore,

$(2a)^3 + (3b)^3 + (4c)^3 - 3(2a)(3b)(4c) = 0$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 72abc$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 72 \times 12 \times 8 \times 6$

$\Rightarrow 8a^3 + 27b^3 + 64c^3 = 41472$

**Factorisation of Algebraic Expressions Using Identities for the Sum and Difference of Two Cubes**

Let us start with a simple question.

What is the value of $(203)^3$?

Yes, its value is 8365427.

For sure, you would have used the identity $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

We can write $(203)^3$ as $(200 + 3)^3$ and then using the identity for $(a + b)^3$, we can find its value as this is an easier method as compared to multiplication.

However, we have many more applications of this identity. We can also factorise algebraic expressions using this identity.

For this, we have to rewrite the identity as follows:

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$
$$\Rightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$\Rightarrow a^3 + b^3 = (a+b)\left[(a+b)^2 - 3ab\right]$$
$$\Rightarrow a^3 + b^3 = (a+b)\left[a^2 + b^2 + 2ab - 3ab\right]$$
$$\Rightarrow \boxed{a^3 + b^3 = (a+b)(a^2 + b^2 - ab)} \qquad ...(1)$$

This form of the identity is used to factorise the expressions of the form $a^3 + b^3$.

In the same way, we can write the identity for $a^3 - b^3$ as follows:

$$\boxed{a^3 - b^3 = (a-b)(a^2 + b^2 + ab)} \qquad ...(2)$$

To understand how to use these identities to factorise expressions, let us see an example.

Let us factorise the expression $x^6 - 729y^6$.

$x^6 - 729y^6$

$= (x^3)^2 - (27y^3)^2$

$= (x^3 + 27y^3)(x^3 - 27y^3)$ [Using $a^2 - b^2 = (a + b)(a - b)$]

$= [(x)^3 + (3y)^3][(x)^3 - (3y)^3]$

Using identities (1) and (2), we obtain

$\Rightarrow (x + 3y)(x^2 + 9y^2 - 3xy)(x - 3y)(x^2 + 9y^2 + 3xy)$

This is the factorised form of the given expression.

To understand this method more clearly, let us solve some more examples.

**Example 1:**

Factorise the expression: $125a^6 - 343$

**Solution:**

$125a^6 - 343 = (5a^2)^3 - (7)^3$

$= (5a^2 - 7)(25a^4 + 49 + 35a^2)$ [Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$]

This is the factorised form of the given expression.

**Example 2:**

Factorise the expression: $x^3 + 64y^3 + 12x^2y + 48xy^2 - (4x + y)^3$

**Solution:**

$x^3 + 64y^3 + 12x^2y + 48xy^2 - (4x + y)^3$

$= x^3 + (4y)^3 + 3(x)(4y)(x + 4y) - (4x + y)^3$

$= (x + 4y)^3 - (4x + y)^3$ [Using $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$]

$= (x + 4y - 4x - y)[(x + 4y)^2 + (4x + y)^2 + (x + 4y)(4x + y)]$

[Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$]

$= (-3x + 3y)[x^2 + 16y^2 + 8xy + 16x^2 + y^2 + 8xy + (4x^2 + 17xy + 4y^2)$

$= 3\,(y - x)\,[21x^2 + 21y^2 + 33xy]$

$= 3\,(y - x) \times 3\,(7x^2 + 7y^2 + 11xy)$

$= 9\,(y - x)\,(7x^2 + 7y^2 + 11xy)$

This is the factorised form of the given expression.

**Example 3:**

Factorise the following expressions.

(1) $x^3 - \frac{1}{x^3}$

(2) $343a^3 - \frac{1}{27a^3}$

(3) $x^3 + \frac{1}{x^3}$

(4) $216m^3 + \frac{64}{m^3}$

**Solution:**

(1)

$$x^3-\frac{1}{x^3}$$

$$=(x)^3-\left(\frac{1}{x}\right)^3$$

$$=\left(x-\frac{1}{x}\right)\left(x^2+x.\frac{1}{x}+\frac{1}{x^2}\right) \qquad \left[\text{Using } a^3-b^3=(a-b)\left(a^2+b^2+ab\right)\right]$$

$$=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)$$

(2)

$$343a^3-\frac{1}{27a^3}$$

$$=(7a)^3-\frac{1}{(3a)^3}$$

$$=\left(7a-\frac{1}{3a}\right)\left\{(7a)^2+7a.\frac{1}{3a}+\frac{1}{(3a)^2}\right\} \qquad \left[\text{Using } a^3-b^3=(a-b)\left(a^2+b^2+ab\right)\right]$$

$$=\left(7a-\frac{1}{3a}\right)\left(49a^2+\frac{7}{3}+\frac{1}{9a^2}\right)$$

(3)

$$x^3+\frac{1}{x^3}$$

$$=(x)^3+\left(\frac{1}{x}\right)^3$$

$$=\left(x+\frac{1}{x}\right)\left(x^2-x.\frac{1}{x}+\frac{1}{x^2}\right) \qquad \left[\text{Using } a^3+b^3=(a+b)\ \left(a^2+b^2-ab\right)\right]$$

$$=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$$

(4)

$$216m^3+\frac{64}{m^3}$$

$$=(6m)^3+\left(\frac{4}{m}\right)^3$$

$$=\left(6m+\frac{4}{m}\right)\left\{(6m)^2-6m.\frac{1}{4m}+\frac{1}{(4m)^2}\right\} \qquad \left[\text{Using } a^3+b^3=(a+b)\ \left(a^2+b^2-ab\right)\right]$$

$$=\left(6m+\frac{4}{m}\right)\left(36m^2-\frac{3}{2}+\frac{1}{16m^2}\right)$$