Mathematics [Official]

CISCE
Academic Year: 2023-2024
Date & Time: 20th February 2024, 2:00 pm

Duration: 3h **Marks: 70**

SECTION A - 65 MARKS

Q1. In subparts (i) to (x) choose the correct options and in subparts (xi) to (xv), answer's the question's as instructed.

1.1. Let L be a set of all straight lines in a plane. The relation R on L defined as 'perpendicular to' is ______.

1. Symmetric and Transitive
2. Transitive
3. Symmetric
4. Equivalence

Solution

Let L be a set of all straight lines in a plane. The relation R on L defined as 'perpendicular to' is **symmetric**.

Explanation:

The relation is symmetric, meaning that if a line (l) is perpendicular to line (m), then line (m) is also perpendicular to line l.

However, if line (l) is perpendicular to line (m) and line (m) is perpendicular to line (n).

Then, lines 'l' and 'n' are parallel rather than perpendicular, but is parallel.

As a result, the provided relation is only symmetric.

1.2. The order and degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right] = \frac{d^2y}{dx^2}$$ are ______.

1. $2, \frac{3}{2}$

2. 2, 3

3. 2, 1

4. 3, 4

Solution

The order and degree of the differential equation

$\left[1+\left(\frac{dy}{dx}\right)^2\right] = \frac{d^2y}{dx^2}$ are **2, 1**.

Explanation:

The given differential equation is

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}$$

Here, the highest derivative is 2

∴ Order = 2 and the power of the highest derivative is 1.

∴ Degree = 1.

1.3. Let A be a non-empty set.

Statement 1: Identity relation on A is Reflexive.

Statement 2: Every Reflexive relation on A is an Identity relation.

1. Both the statements are true.
2. Both the statements are false.
3. Statement 1 is true and Statement 2 is false.
4. Statement 1 is false and Statement 2 is true.

Solution

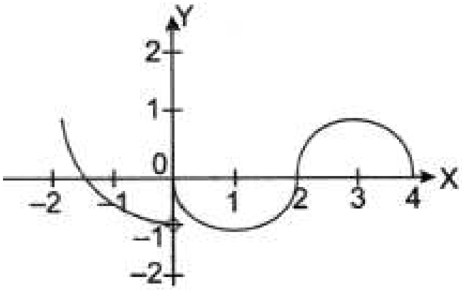
Statement 1 is true and Statement 2 is false.

Explanation:

Consider A = {a, b, c} and define a relation R as R = {(a, a), (b, b), (c, c), (a, b)}.

Then R is a reflexive relation on A, but not an identity relation, because R contains the elements (a, b).

1.4. The graph of the function f is shown below.



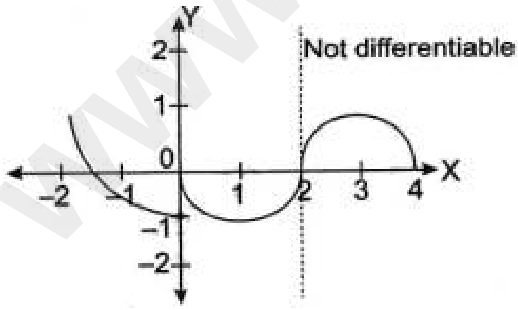
Of the following options, at what values of x is the function f NOT differentiable?

1. At x = 0 and x = 2
2. At x = 1 and x = 3
3. At x = – 1 and x = 1
4. At x = – 1.5 and x = 1.5

Solution

At x = 0 and x = 2

Explanation:



Hence, x = 0 and x = 2, the function f is not differentiable.

1.5.

The value of cosec $\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] - \sec\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$ is equal to _____.

1. –4
2. 0
3. –1
4. 4

Solution

The value of cosec $\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] - \sec\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$ is equal to **0**.

Explanation:

cosec $\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] - \sec\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$

= cosec $\left[-\sin^{-1}\left(\frac{1}{2}\right)\right] - \sec\left[\pi - \cos^{-1}\left(\frac{-1}{2}\right)\right]$

= cosec $\left[-\frac{\pi}{6}\right] - \sec\left[\pi - \frac{\pi}{3}\right]$

= $-$cosec $\left[\frac{\pi}{6}\right] - \sec\left[\frac{2\pi}{3}\right]$

= – cosec 30° – sec 120°

= – cosec 30° – sec [(90° + 30°)]

= – 2 – [– cosec 30°]

= – 2 + cosec 30°

= – 2 + 2

= 0

1.6.

$\int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$ is equal to ______.

1. $\frac{\pi}{12}$
2. $\frac{\pi}{6}$
3. $\frac{\pi}{4}$
4. $\frac{\pi}{3}$
5. $\frac{\pi}{2}$
6. $\frac{2\pi}{3}$

Solution

$\int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$ is equal to $\underline{\mathbf{\frac{\pi}{12}}}$.

Explanation:

Let I = $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2}$

= $\left[\tan^{-1} x\right]_{1}^{\sqrt{3}}$

= $\tan^{-1}\sqrt{3} - \tan^{-1} 1$

= $\frac{\pi}{3} - \frac{\pi}{4}$

= $\frac{4\pi - 3\pi}{12}$

= $\frac{\pi}{12}$

1.7. Assertion: Let the matrices

A = $\begin{pmatrix} -3 & 2 \\ -5 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}$

be such that $A^{100}B = BA^{100}$

Reason: AB = BA implies AB = BA for all positive integers n.

1. **Both Assertion and Reason are true and Reason is the correct explanation for Assertion.**
2. Both Assertion and Reason are true but Reason is not the correct explanation for Assertion.
3. Assertion is true and Reason is false.
4. Assertion is false and Reason is true.

Solution

Both Assertion and Reason are true and Reason is the correct explanation for Assertion.

Explanation:

We have, A = $\begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix}$

Now, AB = $\begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

And BA = $\begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

Hence, AB = BA

Now, A^2 = $\begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix}$

So, A^2B = $\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 10 & -8 \end{bmatrix}$

And BA^2 = $\begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 10 & -8 \end{bmatrix}$

Hence, $A^2B = BA^2$

If, AB = BA and $A^2B = BA^2$...............

Therefore, $A^nB = BA^n$

Also, $A^{100}B = BA^{100}$

Hence, Assertion and Reason both are true.

1.8. If $\int(\cot x - \text{cosec}^2 x)e^x\,dx = e^x f(x) + c$ then f(x) will be ______.

1. cot x + cosec x
2. $\cot^2 x$
3. cot x
4. cosec x

Solution

If $\int(\cot x - \text{cosec}^2 x)e^x dx = e^x f(x) + c$ then f(x) will be **cot x**.

Explanation:

$\int(\cot x - \text{cosec}^2 x)e^x\,dx = e^x f(x) + c$

Then, $\int(\cot x - \text{cosec}^2 x)e^x\,dx$

$= \int e^x \cot x\,dx - \int e^x \text{cosec}^2 x\,dx$

On integrating by parts

$$= \cot x \int e^x dx - \int \frac{d}{dx}\cot x \int e^x dx - \int e^x \text{cosec}^2 dx + c$$

$= e^x \cot x + \int e^x \text{cosec}^2 x\,dx - \int e^x \text{cosec}^2\,dx + c$

$= e^x \cot x + c$

Hence, f(x) = cot x.

1.9. In which one of the following intervals is the function $f(x) = x^3 - 12x$ increasing?

1. (– 2, 2)
2. (– ∞, – 2) ∪ (2, ∞)
3. (– 2, ∞)

4. $(-\infty, 2)$

Solution

$(-\infty, -2) \cup (2, \infty)$

Explanation:

Here, $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

For increasing $f'(x) > 0$

$\therefore 3x^2 - 12 > 0$

$3(x^2 - 4) > 0$

$3(x - 2)(x + 2) > 0$

$\therefore x > 2, -2$

$\therefore$ For $x = -3$

$f'(-3) = 3(-3)^2 - 12$

$= 15 > 0$

For $x = 3$

$f'(3) = 3(3)^2 - 12$

$= 15 > 0$

$\therefore f(x)$ is increasing in interval $(-\infty, -2) \cup (2, \infty)$

1.10. If A and B are symmetric matrices of the same order, then AB – BA is ______.

1. Skew – symmetric matrix
2. Symmetric matrix
3. Diagonal matrix
4. Identity matrix

Solution

If A and B are symmetric matrices of the same order, then AB – BA is **skew – symmetric matrix**.

Explanation:

Given that A and B are symmetric matrices,

A = A′ and B = B′

Then (AB – BA)′ = (AB)′ – (BA)′

= B′A′ – A′B′ …[∵ A = A′, B = B′]

= BA – AB

= – (AB – BA)

Thus, it is skew – symmetric.

1.11.

Find the derivative of $y = \log x + \frac{1}{x}$ with respect to x.

Solution

$$y = \log x + \frac{1}{x}$$

On differentiating both sides, w.r.t. x

$$\frac{dy}{dx} = \frac{d(\log x)}{dx} + \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x} + \left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x} - \frac{1}{x^2}$$

1.12. Teena is practising for an upcoming Rifle Shooting tournament. The probability of her shooting the target in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} shots are 0.4, 0.3, 0.2 and 0.1 respectively. Find the probability of at least one shot of Teena hitting the target.

Solution

Required probability

= 1 – Probability that neither of the target hits

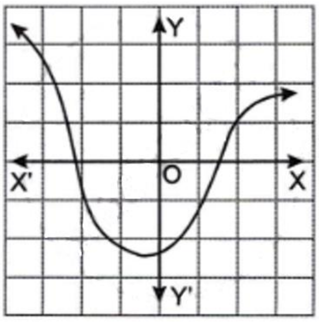
= 1 – [(1 – 0.4) (1 – 0.3) (1 – 0.2) (1 – 0.1)]

$= 1 - [0.6 \times 0.7 \times 0.8 \times 0.9]$

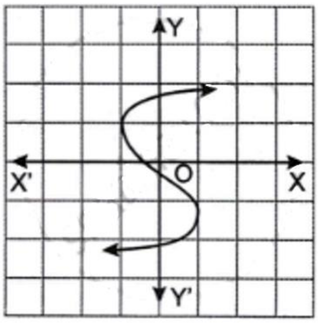
$= 1 - 0.3024$

$= 0.6976$

1.13. Which one of the following graphs is a function of x?



Graph A



Graph B

Solution

Graph A represents the function of x.

In graph B, we see that for a value of x, there are 2 values of y.

1.14. Evaluate:

$$\int_0^6 |x+3|dx$$

Solution

Let I = $\int_0^6 |x+3|dx$

As, $0 \le x \le 6$

$\Rightarrow -3 \le x + 3 \le 9$

$x + 3 > 0$

$\Rightarrow |x + 3| = |x + 3|$

$$\therefore \int_0^6 |x+3|dx = \int_0^6 (x+3)dx$$

$= \left[\frac{x^2}{2} + 3x\right]_0^6$

$= \left(\frac{6^2}{2} + 3 \times 6\right) - 0$

= 18 + 18

= 36

1.15. Given that

$\frac{1}{y} + \frac{1}{x} = \frac{1}{12}$ and y

decreases at a rate of 1 cms^{-1}, find the rate of change of x when x = 5 cm and y = 1 cm.

Solution

Given, $\frac{dy}{dt}$ = 1 cm/s

And $\frac{1}{y} + \frac{1}{x} = \frac{1}{12}$

On differentiating w.r.t 't' on both sides, we get

$$\frac{-1}{y^2}\frac{dy}{dt} - \frac{1}{x^2}\frac{dx}{dt} = 0$$

Put x = 5, y = 1 and $\frac{dy}{dt} = 1$

$$\therefore \frac{-1}{1^2} \times 1 - \frac{1}{5^2} \times \frac{dx}{dt} = 0$$

$$\Rightarrow -1 - \frac{1}{25}\frac{dx}{dt} = 1$$

$$\Rightarrow \frac{-1}{25}\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = -25$$

Hence, the rate of change of 'x' is – 25 cm/sec.

Q2.

2.1.

Let $f: R\left\{\frac{-1}{3}\right\} \rightarrow R-\{0\}$ be defined as $f(x)=\frac{5}{3x+1}$ is invertible. Find $f^{-1}(x)$.

Solution

Given, $f(x)=\frac{5}{3x+1}$ and is invertible.

So, we must check for invertibility.

Now, let f(x) = y = $\frac{5}{3x+1}$

$\Rightarrow$ y(3x + 1) = 5

$\Rightarrow$ 3xy + y = 5

$\Rightarrow$ 3xy = 5 – y

$\Rightarrow x=\frac{5-y}{3y}$

$\therefore \; f^{-1}(y)=\frac{5-y}{3y}$

Now put y = x

$\Rightarrow f^{-1}(x)=\frac{5-x}{3x}$

OR

2.2.

If f : R $\rightarrow$ R is defined by $f(x)=\frac{2x-7}{4}$, show that f(x) is one-one and onto.

Solution

Given, $f(x) = \dfrac{2x-7}{4}$

For one-one

Let x, $x_2 \in R$

$f(x_1) = f(x_2)$

$$\frac{2x_1-7}{4} = \frac{2x_2-7}{4}$$

$\Rightarrow 2x_1 - 7 = 2x_2 - 7$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

So, f(x) is a one-to-one function.

For onto

Put $y = \dfrac{2x-7}{4}$

$\Rightarrow 4y = 2x - 7$

$\Rightarrow 4y + 7 = 2x$

$\Rightarrow x = \dfrac{4y+7}{2}$ $\forall$ y ∈ R ∃ a unique x ∈ R

Therefore, f(x) is onto.

Q3. Find the value of the determinant given below, without expanding it at any stage.

$$\begin{vmatrix} \beta\gamma & 1 & \alpha(\beta+\gamma) \\ \gamma\alpha & 1 & \beta(\gamma+\alpha) \\ \alpha\beta & 1 & \gamma(\alpha+\beta) \end{vmatrix}$$

Solution

Given: $\begin{vmatrix} \beta\gamma & 1 & \alpha(\beta+\gamma) \\ \gamma\alpha & 1 & \beta(\gamma+\alpha) \\ \alpha\beta & 1 & \gamma(\alpha+\beta) \end{vmatrix}$

= $\begin{vmatrix} \beta\gamma & 1 & \alpha\beta+\alpha\gamma \\ \gamma\alpha & 1 & \beta\gamma+\beta\alpha \\ \alpha\beta & 1 & \gamma\alpha+\gamma\beta \end{vmatrix}$

Applying $C_3 \rightarrow C_1 + C_3$

= $\begin{vmatrix} \beta\gamma & 1 & \alpha\beta+\beta\gamma+\alpha\gamma \\ \gamma\alpha & 1 & \beta\gamma+\beta\alpha+\gamma\alpha \\ \alpha\beta & 1 & \alpha\beta+\gamma\beta+\gamma\alpha \end{vmatrix}$

= $(\alpha\beta+\beta\gamma+\gamma\alpha)\begin{vmatrix} \beta\gamma & 1 & 1 \\ \gamma\alpha & 1 & 1 \\ \alpha\beta & 1 & 1 \end{vmatrix}$

= $(\alpha\beta + \beta\gamma + \gamma\alpha) \times 0$ ...($\because$ C_2 and C_3 are similar)

= 0

Q4.

4.1. Determine the value of 'k' for which the following function is continuous at x = 3

$$f(x) = \begin{cases} \frac{(x+3)^2-36}{x-3} & x \neq 3 \\ k & x = 3 \end{cases}$$

Solution

LHL = $\lim\limits_{x\to 3^-} f(x)$

= $\lim\limits_{x\to 3^-} \left(\dfrac{(x+3)^2-36}{x-3}\right)$

= $\lim\limits_{h\to 0} \left(\dfrac{(3-h+3)^2-36}{3-h-3}\right)$

Put x = 3 – h

$$= \lim_{h \to 0} \left(\frac{(6-h)^2 - 36}{-h} \right)$$

$$= \lim_{h \to 0} \left(\frac{36 + h^2 - 12h - 36}{-h} \right)$$

$$= \lim_{h \to 0} \left(\frac{h^2 - 12h}{-h} \right)$$

$$= \lim_{h \to 0} (12 - h)$$

= 12

RHL = $\lim_{x \to 3^+} f(x)$

$$= \lim_{x \to 3^+} \left[\frac{(x+3)^2 - 36}{x - 3} \right]$$

$$= \lim_{h \to 0} \left[\frac{(3+h+3)^2 - 36}{3 + h - 3} \right]$$

Put x = 3 + h

$$= \lim_{h \to 0} \left[\frac{(6+h)^2 - 36}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{36 + h^2 + 12h - 36}{h} \right]$$

$$= \lim_{h \to 0} (h + 12)$$

= 12

For continuity,

LHL = RHL = f(3)

= 12 = 12 = k

Hence, k = 12

OR

4.2. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Solution

If a tangent is parallel to the chord joining the points (2, 0) and (4, 4), then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2} = \frac{4}{2} = 2.$

Now, the slope of the tangents to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$2(x - 2) = 2$

$\Rightarrow x - 2 = 1$

$\Rightarrow x = 3$

When $x = 3$, $y = (3 - 2)^2 = 1$.

Hence, the required point is (3, 1).

Q5. Evaluate:

$$\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$$

Solution

Let I = $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ ...(i)

Applying property,

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

I = $\int_0^{2\pi} \frac{dx}{1+e^{\sin(2\pi-x)}}$

= $\int_0^{2\pi} \frac{dx}{1+e^{-\sin x}}$

= $\int_0^{2\pi} \frac{dx}{1+\frac{1}{e^{\sin x}}}$

= $\int_0^{2\pi} \frac{e^{\sin x}dx}{e^{\sin x}+1}$ ...(ii)

On adding equations (i) and (ii), we get

2I = $\int_0^{2\pi} \frac{dx}{1+e^{\sin x}} + \int_0^{2\pi} \frac{e^{\sin x}dx}{1+e^{\sin x}}$

= $\int_0^{2\pi} \left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$

= $\int_0^{2\pi} 1.\,dx$

⇒ 2I = $[x]_0^{2\pi}$

⇒ 2I = [2π]

⇒ I = π

Q6.

Evaluate P(A ∪ B), if 2P(A) = P(B) = $\frac{5}{13}$ and P(A | B) = $\frac{2}{5}$

Solution

It is given that,

2P(A) = P(B) = $\frac{5}{13}$

⇒ P(A) = $\frac{5}{26}$ and P(B) = $\frac{5}{13}$

$\Rightarrow$ P(A | B) = $\frac{2}{5}$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{2}{5}$$

$$\Rightarrow P(A \cap B) = \frac{2}{5} \times P(B)$$

= $\frac{2}{5} \times \frac{5}{13}$

= $\frac{2}{13}$

It is known that,

P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

$\Rightarrow$ P(A ∪ B) = $\frac{5}{26} + \frac{5}{13} - \frac{2}{13}$

$\Rightarrow$ P(A ∪ B) = $\frac{5+10-4}{26}$

$\Rightarrow$ P(A ∪ B) = $\frac{11}{26}$

Q7.

If y = 3 cos(log x) + 4 sin(log x), show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution

y = 3 cos (log x) + 4 sin (log x)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -3\sin(\log x) \times \frac{1}{x} + 4\cos(\log x) \times \frac{1}{x}$$

$$x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Again differentiating both sides w.r.t x, we get

$$x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right) = -3\cos(\log x) \times \frac{1}{x} - 4\sin(\log x) \times \frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Hence proved

Q8.

8.1.

Solve for x: $\sin^{-1}\left(\frac{x}{2}\right) + \cos^{-1}x = \frac{\pi}{6}$

Solution

$$\sin^{-1}\left(\frac{x}{2}\right) + \cos^{-1}x = \frac{\pi}{6}$$

Since, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$$\therefore \sin^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow -\sin^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6} - \frac{\pi}{2}$$

$$\Rightarrow -\sin^{-1}x + \sin^{-1}\frac{x}{2} = \frac{2\pi - 6\pi}{12}$$

$$\Rightarrow -\sin^{-1}x + \sin^{-1}\frac{x}{2} = -\frac{\pi}{3}$$

$$\Rightarrow \sin^{-1}\frac{x}{2} = -\frac{\pi}{3} + \sin^{-1}x$$

$$\Rightarrow \frac{x}{2} = \sin\left(-\frac{\pi}{3} + \sin^{-1} x\right)$$

$$\Rightarrow \frac{x}{2} = \sin\left(\frac{-\pi}{3}\right)\cos\left(\sin^{-1} x\right) + \cos\left(\frac{-\pi}{3}\right)\sin\left(\sin^{-1} x\right)$$

$$\Rightarrow \frac{x}{2} = -\sin\ \frac{\pi}{3}\cos\cos^{-1}\sqrt{1-x^2} + \cos\left(\frac{\pi}{3}\right)x$$

$$\Rightarrow \frac{x}{2} = -\frac{\sqrt{3}}{2}\sqrt{1-x^2} + \frac{x}{2}$$

$$\Rightarrow 0 = -\frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$\Rightarrow 1 - x^2 = 0$

$\Rightarrow x^2 = 1$

∴ x = 1 is the only answer because x = – 1 will not satisfy above question.

OR

8.2. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, show that $x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0$

Solution

Given, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}z$

$$\Rightarrow \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] = \left(\pi - \sin^{-1} z\right)$$

$$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\left(\pi - \sin^{-1} z\right)$$

$$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = z$$

$$\Rightarrow x\sqrt{1-y^2} = z - y\sqrt{1-x^2}$$

Now squaring on both sides, we get,

$$\left(x\sqrt{1-y^2}\right)^2 = \left(z - y\sqrt{1-x^2}\right)^2$$

$$\Rightarrow x^2(1-y^2) = \left(z^2 + y^2(1-x^2) - 2zy\sqrt{1-x^2}\right)$$

$$\Rightarrow x^2 - x^2y^2 = z^2 + y^2 - x^2y^2 - 2yz\sqrt{1-x^2}$$

$$\Rightarrow x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0$$

Hence proved

Q9.

9.1. Evaluate:

$$\int x^2 \cos x \, dx$$

Solution

Let I = $\int x^2 \cos x \, dx$

On applying integration by parts

I = $x^2 \int \cos x \, dx - \int \left\{ \frac{d}{dx}(x^2) \int \cos x . dx \right\} dx$

I = $x^2 \sin x - \int 2x \sin x \, dx$

Again on applying integration by parts

= x^2 sin x – 2[– x cos x – ∫ – cos x dx]

= x^2 sin x – 2[– x cos x + sin x + c]

= x^2 sin x + 2x cos x – 2 sin x + c

= (x^2 – 2) sin x + 2x cos x + c

OR

9.2. Evaluate:

$$\int \frac{x+7}{x^2+4x+7} dx$$

Solution

Let I = $\int \frac{x+7}{x^2+4x+7}dx$

On applying partial integration method

$x+7 = \text{A}\frac{d}{dx}\left(x^2+4x+7\right)+\text{B}$

x + 7 = A(2x + 4) + B

Then, A = $\frac{1}{2}$ and B = 5

Then, I = $\int \frac{\frac{1}{2}(2x+4)+5}{x^2+4x+7}dx$

= $\frac{1}{2}\int \frac{(2x+4)}{x^2+4x+7}dx + 5\int \frac{1}{(x^2+4x+7)}dx$

= $\frac{1}{2}\log\left|x^2+4x+7\right| + 5\int \frac{1}{(x+2)^2+\left(\sqrt{3}\right)^2}dx + c$

= $\frac{1}{2}\log\left|x^2+4x+7\right| + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c$

Q10. A jewellery seller has precious gems in white and red colour which he has put in three boxes.

The distribution of these gems is shown in the table given below:

Box	Number of Gems	
	White	Red
I	1	2
I	2	3
III	3	1

He wants to gift two gems to his mother. So, he asks her to select one box at random and pick out any two gems one after the other without replacement from the selected box. The mother selects one white and one red gem.

Calculate the probability that the gems drawn are from Box II.

Solution

The probability of selecting each box is:

$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Two gems are chosen from the selected box.

Let A be the event where one white and one red gem are chosen.

$P(A \mid E_1)$ = Probability of drawing 1 white and 1 red gem when box I is chosen

$$= \frac{{}^1C_1 \times {}^2C_1}{{}^3C_2}$$

$$= \frac{2}{3}$$

$P(A \mid E_2)$ = Probability of drawing 1 white and 1 red gem when box II is chosen

$$= \frac{{}^2C_1 \times {}^3C_1}{{}^5C_2}$$

$$= \frac{2 \times 3}{10}$$

$$= \frac{3}{5}$$

$P(A \mid E_3)$ = Probability of drawing 1 white and 1 red gem when box III is chosen

$$= \frac{{}^3C_1 \times {}^1C_1}{{}^4C_2}$$

$$= \frac{3 \times 1}{6}$$

$$= \frac{1}{2}$$

According to Bayes theorem, we have

$$P(E_2 \mid A)=\frac{P(E_2)P(A \mid E_2)}{P(E_1)P(A \mid E_1)+P(E_2)P(A \mid E_2)+P(E_3)P(A \mid E_3)}$$

$$=\frac{\frac{1}{3}\times\frac{3}{5}}{\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{1}{2}}$$

$$=\frac{\frac{1}{5}}{\frac{2}{9}+\frac{1}{5}+\frac{1}{6}}$$

$$=\frac{\frac{1}{5}}{\frac{20+18+15}{90}}$$

$$=\frac{90}{5\times 53}$$

$$=\frac{18}{53}$$

Q11. A furniture factory uses three types of wood namely, teakwood, rosewood and satinwood for manufacturing three types of furniture, that are, table, chair and cot.

The wood requirements (in tonnes) for each type of furniture are given below:

	Table	**Chair**	**Cot**
Teakwood	2	3	4
Rosewood	1	1	2
Satinwood	3	2	1

It is found that 29 tonnes of teakwood, 13 tonnes of rosewood and 16 tonnes of satinwood are available to make all three types of furniture.

Using the above information, answer the following questions:

i. Express the data given in the table above in the form of a set of simultaneous equations.

ii. Solve the set of simultaneous equations formed in subpart (i) by matrix method.

iii. Hence, find the number of table(s), chair(s) and cot(s) produced.

Solution

Let the number of tables, chairs and cots produced be x, y and z.

i. Then, the system of simultaneous equations produced is:

$2x + 3y + 4z = 29$

$x + y + 2z = 13$

$3x + 2y + z = 16$

ii. Part (i) equations are in matrix form as follows:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

i.e., AX = B

$\Rightarrow X = A^{-1}B$

$|A| = 2(1 - 4) - 3(1 - 6) + 4(2 - 3)$

$= -6 + 15 - 4$

$= 15 - 10$

$= 5 \neq 0$

As a result, the inverse exists.

Then, $a_{11} = (-1)^{1+1}(1 - 4) = -3$

$a_{12} = (-1)^{1+2}(1 - 6) = 5$

$a_{13} = (-1)^{1+3}(2 - 3) = -1$

$a_{21} = (-1)^{2+1}(3 - 8) = 5$

$a_{22} = (-1)^{2+2}(2 - 12) = -10$

$a_{23} = (-1)^{2+3}(4 - 9) = 5$

$a_{31} = (-1)^{3+1}(6 - 4) = 2$

$a_{32} = (-1)^{3+2}(4 - 4) = 0$

$a_{33} = (-1)^{3+3}(2 - 3) = -1$

$$\text{adj A} = \begin{bmatrix} -3 & 5 & -1 \\ 5 & -10 & 5 \\ 2 & 0 & -1 \end{bmatrix}^1 = \begin{bmatrix} -3 & 5 & 2 \\ 5 & -10 & 0 \\ -1 & 5 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{[adjA]}{|A|} = \frac{1}{5}\begin{bmatrix} -3 & 5 & 2 \\ 5 & -10 & 0 \\ -1 & 5 & -1 \end{bmatrix}$$

$\therefore X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 5 & 2 \\ 5 & -10 & 0 \\ -1 & 5 & -1 \end{bmatrix}\begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -87 + 65 + 32 \\ 145 - 130 + 0 \\ -29 + 65 - 16 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Hence, x = 2, y = 3, z = 4

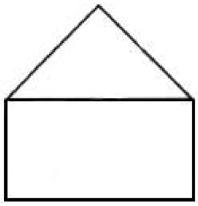
iii. ∴ Number of table(s) produced = 2

Number of chair(s) produced = 3

Number of cot(s) produced = 4

Q12.

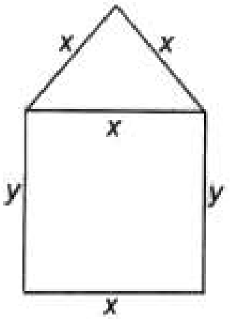
12.1. Mrs. Roy designs a window in her son's study room so that the room gets maximum sunlight. She designs the window in the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the window that will admit maximum sunlight into the room.



Solution

Let x and y be the window dimensions and x be the side of the equilateral portion.

Let A be the complete area of the window (through which light enters):



A = $xy+\frac{\sqrt{3}}{4}x^2$

Also, x + 2y + 2x = 12 ...(Given)

$\Rightarrow$ 3x + 2y = 12

$\Rightarrow y=\frac{12-3x}{2}$

Then, A = $x\times\left(\frac{12-3x}{2}\right)+\frac{\sqrt{3}}{4}x^2$

= $6x-\frac{3x^2}{2}+\frac{\sqrt{3}}{4}x^2$

Then, $\frac{dA}{dx}=6-3x+\frac{\sqrt{3}}{2}x$

For maximum light to enter, the area of the window should be the maximum

Put $\frac{dA}{dx}=0$

$$6-3x+\frac{\sqrt{3}}{2}x=0$$

$$x = \frac{12}{6-\sqrt{3}}$$

Again, $\frac{d^2A}{dx^2} = -3 + \frac{\sqrt{3}}{2} < 0$ ...(For any value of x)

i.e., A is maximum if $x = \frac{12}{6-\sqrt{3}}$ and

$$y = \frac{12 - 3\left(\frac{12}{6-\sqrt{3}}\right)}{2}$$

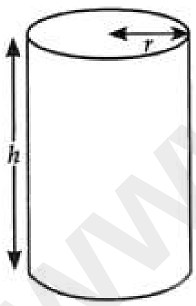
$$= \frac{18 - 6\sqrt{3}}{6-\sqrt{3}}$$

Hence dimensions are $\left(\frac{12}{6-\sqrt{3}}\right)m.$

and $\left(\frac{18-6\sqrt{3}}{6-\sqrt{3}}\right)m.$

OR

12.2. Sumit has bought a closed cylindrical dustbin. The radius of the dustbin is 'r' cm and height is 'h' cm. It has a volume of 20π cm^3.



a. Express 'h' in terms of 'r', using the given volume.

b. Prove that the total surface area of the dustbin is $2\pi r^2 + \frac{40\pi}{r}$

c. Sumit wants to paint the dustbin. The cost of painting the base and top of the dustbin is ₹ 2 per cm^2 and the cost of painting the curved side is ₹ 25 per cm^2. Find the total cost in terms of 'r', for painting the outer surface of the dustbin including the base and top.

d. Calculate the minimum cost for painting the dustbin.

Solution

Given, radius of dustbin is r and height is h.

a. Volume of dustbin,

$V = 20\pi\ cm^3$

Then, $\pi r^2 h = 20\pi$

$\Rightarrow r^2 h = 20$

$\Rightarrow h = \frac{20}{r^2}$

b. T.S.A. of dustbin = C.S.A + 2 base area

$= 2\pi rh + 2\pi r^2$

$= 2\pi r \times \frac{20}{r^2} + 2\pi r^2$

$= \frac{40\pi}{r} + 2\pi r^2$

$= 2\pi r^2 + \frac{40\pi}{r}$

Hence proved

c. C.S.A. of dustbin = $\frac{40\pi}{r} cm^2$

Then, cost of painting (SA) = $\frac{40\pi}{r} \times 25$

= ₹ $\frac{1000\pi}{r}$

Base and top area of dustbin = $2\pi r^2$

Then, cost of painting (top and bottom)

$= 2\pi r^2 \times 2$

= ₹ $4\pi r^2$

Total cost of painting = ₹ $\left(\frac{1000\pi}{r} + 4\pi r^2\right)$

d. Cost of painting,

$$C = \frac{1000\pi}{r} + 4\pi r^2$$

$$\therefore \frac{dC}{dr} = -\frac{1000\pi}{r^2} + 8\pi r$$

For minimum cost,

Put $\frac{dC}{dr} = 0$

$$-\frac{1000\pi}{r^2} + 8\pi r = 0$$

$$\Rightarrow 8\pi r = \frac{1000\pi}{r^2}$$

$\Rightarrow r^3 = 125$

$\Rightarrow r = 5$ cm

And $\frac{d^2C}{dr^2} = \frac{2000\pi}{r^3} + 8\pi > 0$

Then, $\frac{d^2C}{dr^2}$ is minimum for r = 5 cm.

Then, minimum cost of painting = $\frac{1000\pi}{5} + 8\pi(5)$

= 200π + 40π

= ₹ 240π

Q13.

13.1. Find the particular solution of the differential equation:

$2y\, e^{x/y}\, dx + (y - 2x\, e^{x/y})\, dy = 0$ given that x = 0 when y = 1.

Solution

$$2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{2xe^{\frac{x}{y}-y}}{2ye^{\frac{x}{y}}}$$

Given differential equation is a homogeneous differential equation.

∴ Put x = vy

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow 2e^v dv = -\frac{1}{y}dy$$

Integrating on both the sides

$$\Rightarrow 2\int e^v dv = -\int \frac{1}{y}dy$$

$$\Rightarrow 2e^v = -\log|y| + \log C$$

$$\Rightarrow 2e^v = \log\left|\frac{c}{y}\right|$$

$$\Rightarrow 2e^{\frac{x}{y}} = \log\left|\frac{c}{y}\right|$$

Given that at x = 0, y = 1

$$2e^0 = \log\left|\frac{c}{1}\right|$$

$\Rightarrow$ C = e^2

$$\therefore 2e^{\frac{x}{y}} = \log\frac{e^2}{y}$$

$$\Rightarrow \log y = -2e^{\frac{x}{y}} + 2$$

$$\Rightarrow y = e^2 - 2e^{\frac{x}{y}}$$

OR

13.2. For the following differential equation, find a particular solution satisfying the given condition:

$x\left(x^2-1\right)\frac{dy}{dx}=1, y=0$ when $x=2$

Solution

We have,

$$x\left(x^2-1\right)\frac{dy}{dx}=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{1}{x\left(x^2-1\right)}$$
$$\Rightarrow dy=\left\{\frac{1}{x\left(x^2-1\right)}\right\}dx$$

Integrating both sides, we get

$$\int dy=\int\left\{\frac{1}{x\left(x^2-1\right)}\right\}dx$$
$$\Rightarrow y=\int\left\{\frac{1}{x\left(x^2-1\right)}\right\}dx+C$$
$$\Rightarrow y=\int\left\{\frac{1}{x\left(x+1\right)\left(x-1\right)}\right\}dx+C\ldots\ldots\ldots(1)$$

Let $\frac{1}{x\left(x+1\right)\left(x-1\right)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$

$$\Rightarrow 1=A\left(x+1\right)\left(x-1\right)+Bx\left(x-1\right)+Cx\left(x+1\right)$$
$$\Rightarrow 1=A\left(x^2-1\right)+B\left(x^2-x\right)+C\left(x^2+x\right)$$
$$\Rightarrow 1=x^2\left(A+B+C\right)+x\left(-B+C\right)-A$$

Comparing both sides, we get

−A = 1.........(2)

−B + C = 0.........(3)

A + B + C = 0........(4)

Solving (2), (3) and (4), we get

A = −1

$B=\frac{1}{2}$

$C=\frac{1}{2}$

$\therefore \frac{1}{x(x+1)(x-1)}=\frac{-1}{x}+\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$

Now, (1) becomes

$$y=\int\left\{\frac{-1}{x}+\frac{1}{2(x+1)}+\frac{1}{2(x-1)}\right\}dx+C$$

$$\Rightarrow y=-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

$$\Rightarrow y=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$$

$$\Rightarrow y=\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|-\log|x|+C$$

Given:- $y(2)=0$

$$\therefore 0=\frac{1}{2}\log|2-1|+\frac{1}{2}\log|2+1|-\log|2|+C$$

$$\Rightarrow C=\log|2|-\frac{1}{2}\log|3|$$

Substituting the value of C, we get

$$y=\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|-\log|x|+\log|2|-\frac{1}{2}\log|3|$$

$$\Rightarrow 2y=\log|x-1|+\log|x+1|-2\log|x|+2\log|2|-\log|3|$$

$$\Rightarrow 2y=\log|x-1|+\log|x+1|-\log\left|x^2\right|+\log|4|-\log|3|$$

$$\Rightarrow 2y=\log\frac{(x-1)(x+1)}{x^2}-(\log|3|-\log|4|)$$

$$\Rightarrow y=\frac{1}{2}\log\frac{\left(x^2-1\right)}{x^2}-\frac{1}{2}\log\left(\frac{3}{4}\right)$$

Q14. A primary school teacher wants to teach the concept of 'larger number' to the students of Class II.

To teach this concept, he conducts an activity in his class. He asks the children to select two numbers from a set of numbers given as 2, 3, 4, 5 one after the other without replacement.

All the outcomes of this activity are tabulated in the form of ordered pairs given below:

	2	3	4	5
2	(2, 2)	(2, 3)	(2, 4)	
3	(3, 2)	(3, 3)		(3, 5)
4	(4, 2)		(4, 4)	(4, 5)
5		(5, 3)	(5, 4)	(5, 5)

i. Complete the table given above.

ii. Find the total number of ordered pairs having one larger number.

iii. Let the random variable X denote the larger of two numbers in the ordered pair.
Now, complete the probability distribution table for X given below.

X	3	4	5
P(X = x)			

i. Find the value of $P(X < 5)$

ii. Calculate the expected value of the probability distribution.

Solution

i.

	2	3	4	5
2	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 2)	(4, 3)	(4, 4)	(4, 5)
5	(5, 2)	(5, 3)	(5, 4)	(5, 5)

ii. Total number of ordered pairings with one greater number = 12 iii.

X	3	4	5
P(X = x)	212	412	

iv. P(X < 5) = P(3) + P(4)

$$\frac{2}{12}+\frac{4}{12}=\frac{6}{12}$$

P(X < 5) = $\frac{1}{2}$

v. E(x) = ∑xP(x)

$$= 3\times\frac{2}{12}+4\times\frac{4}{12}+5\times\frac{6}{12}$$

$$= \frac{6}{12}+\frac{16}{12}+\frac{30}{12}$$

$$= \frac{52}{12}$$

$$= \frac{13}{3}$$

SECTION B - 15 MARKS

Q15. In subparts (i) and (ii) choose the correct options and in subparts (iii) to (v), answer the questions as instructed.

15.1.

If $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}-3\hat{k}$ then the value of $\left|\vec{a}-2\vec{b}\right|$ will be _____.

1. $\sqrt{85}$
2. **$\sqrt{86}$**
3. $\sqrt{87}$
4. $\sqrt{88}$

Solution

If $\vec{a}=3\hat{i}-2\hat{j}+\widehat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}-3\widehat{k}$ then the value of $\left|\vec{a}-2\vec{b}\right|$ will be $\underline{\sqrt{\mathbf{86}}}$.

Explanation:

Given, $\vec{a}=3\hat{i}-2\hat{j}+\widehat{k}$

$\vec{b}=2\hat{i}-4\hat{j}-3\widehat{k}$

Then, $\left(\vec{a}-2\vec{b}\right)=\left[\left(3\hat{i}-2\hat{j}+\widehat{k}\right)-2\left(2\hat{i}-4\hat{j}-3\widehat{k}\right)\right]$

= $\left[3\hat{i}-2\hat{i}+\widehat{k}-4\hat{i}+8\hat{i}+6\hat{i}\right]$

= $\left(-\hat{i}+6\hat{j}+7\widehat{k}\right)$

$\left|\vec{a}-2\vec{b}\right|=\sqrt{(-1)^2+(6)^2+(7)^2}$

= $\sqrt{1+36+49}$

= $\sqrt{86}$

15.2. If a line makes an angle α, β and γ with positive direction of the coordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ will be ______.

1. 1
2. 3
3. −2
4. 2

Solution

If a line makes an angle α, β and γ with positive direction of the coordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ will be **2**.

Explanation:

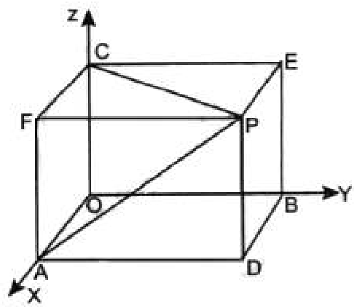
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$

$\Rightarrow 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

15.3. In the figure given below, if the coordinates of the point P are (a, b, c), then what are the perpendicular distances of P from XY, YZ and ZX planes respectively?



Solution

Coordinates of point P(a, b, c)

Then, x - coordinate of P is the perpendicular distance of P from the YZ plane

y - Coordinate of P is the perpendicular distance between P and the XZ plane

z - Coordinate of P is the perpendicular distance of P from the XZ plane.

Perpendicular distance from XY - plane = c

Perpendicular distance from YZ - plane = o

Perpendicular distance from XZ - plane = b

15.4.

If $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} - 3\hat{j} + \hat{k}$, find the projection of $\vec{b}$ on $\vec{a}$.

Solution

Given, $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

And $\vec{b} = 5\hat{i} - 3\hat{j} + \hat{k}$

Projection of $\vec{b}$ on $\vec{a} = \frac{1}{|\vec{a}|}\left(\vec{a}\,.\,\vec{b}\right)$

$\left(\vec{a}\,.\,\vec{b}\right)$ = (2 × 5) + (1 × (– 3)) + (2 × 1)

= 10 – 3 + 2

= 9

Magnitude of $\left|\vec{a}\right| = \sqrt{2^2 + 1 + 2^2}$

= $\sqrt{4 + 1 + 4}$

= $\sqrt{9}$

= 3

∴ Required projection = $\frac{9}{3}$ = 3

15.5.

Find a vector of magnitude 20 units parallel to the vector $2\hat{i} + 5\hat{j} + 4\widehat{k}$.

Solution

Given vector, $\vec{a} = 2\hat{i} + 5\hat{j} + 4\widehat{k}$

$\left|\vec{a}\right| = \sqrt{2^2 + 5^2 + 4^2}$

= $\sqrt{4 + 25 + 16}$

= $\sqrt{45}$

= $3\sqrt{5}$

$\therefore \widehat{a} = \frac{\vec{a}}{\left|\vec{a}\right|}$

= $\frac{2\hat{i} + 5\hat{j} + 4\widehat{k}}{3\sqrt{5}}$

Thus, the vector of magnitude 20 units and parallel to the $\vec{a}$ is:

= $\pm 20.\,\widehat{a}$

= $\pm \frac{20\left(2\hat{i} + 5\hat{j} + 4\widehat{k}\right)}{3\sqrt{5}}$

$= \pm\frac{40\hat{i}}{3\sqrt{5}} \pm \frac{100\hat{j}}{3\sqrt{5}} \pm \frac{80\widehat{k}}{3\sqrt{5}}$

$= \pm\frac{8\sqrt{5}}{3}\hat{i} \pm \frac{20\sqrt{5}}{3}\hat{j} \pm \frac{16\sqrt{5}}{3}\widehat{k}$

Q16.

16.1.

If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors, then prove that either $\vec{b} = \vec{c}$ or $\vec{a}$ and $\left(\vec{b} - \vec{c}\right)$ are parallel.

Solution

Given, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$

$\Rightarrow \vec{a} \times \left(\vec{b} - \vec{c}\right) = 0$

Then, $\vec{b} - \vec{c} = 0$, because $\vec{a}$ is a non-zero vector and the cross-product of two vectors is zero when their angle is 0° i.e., they are parallel to each other.

or $\vec{a}$ and $\left(\vec{b} - \vec{c}\right)$ are parallel.

Hence proved

OR

16.2.

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $\left|\vec{a} \times \vec{b}\right| = \vec{a}.\vec{b}$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution

Given: $\left|\vec{a} \times \vec{b}\right| = \vec{a}.\vec{b}$

$\Rightarrow |\vec{a}|\left|\vec{b}\right| \sin\theta = |\vec{a}|\left|\vec{b}\right| \cos\theta$

$\Rightarrow \sin\theta = \cos\theta$

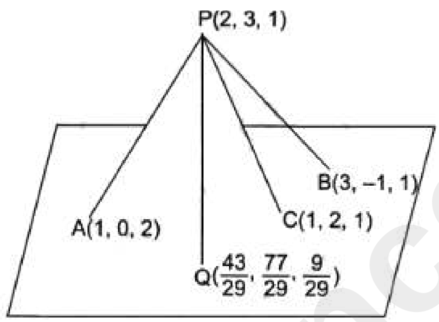
$\Rightarrow \tan\theta = 1$

$\theta = 45°$

Hence, angle is 45°.

Q17. A mobile tower is situated at the top of a hill. Consider the surface on which the tower stands as a plane having points A(1, 0, 2), B(3, –1, 1) and C(1, 2, 1) on it. The mobile tower is tied with three cables from the points A, B and C such that it stands vertically on the ground. The top of the tower is at point P(2, 3, 1) as shown in the figure below. The foot of the perpendicular from the point P on the plane is at the point

$Q\left(\frac{43}{29}, \frac{77}{29}, \frac{9}{29}\right)$.



Answer the following questions.

i. Find the equation of the plane containing the points A, B and C.

ii. Find the equation of the line PQ.

iii. Calculate the height of the tower.

Solution

i. A(1, 0, 2), B(3, –1, 1), C(1, 2, 1)

Equation of plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-0 & z-2 \\ 3-1 & -1-0 & 1-2 \\ 1-1 & 2-0 & 1-2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-0 & z-2 \\ 2 & -1 & -1 \\ 0 & 2 & -1 \end{vmatrix} = 0$$

⇒ (x – 1)(1 + 2) – y(–2 – 0) + (z – 2)(4 – 0) = 0

⇒ 3(x – 1) + 2y + 4(z – 2) = 0

⇒ 3x – 3 + 2y + 4z – 8 = 0

⇒ 3x + 2y + 4z = 11

ii. Let a and b be the position vectors of the points P(2, 3, 1) and $Q\left(\frac{43}{29}, \frac{77}{29}, \frac{9}{29}\right)$ respectively.

Then, $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \frac{43}{29}\hat{i} + \frac{77}{29}\hat{j} + \frac{9}{29}\hat{k}$

Let $\vec{r}$ represent the position vector of any point A(x, y, z) on the line connecting points P and Q. The vector equation for the line is

$\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

$= \left(2\hat{i} + 3\hat{j} + \hat{k}\right) + \lambda\left[\left(\frac{43}{29} - 2\right)\hat{i} + \left(\frac{77}{29} - 3\right)\hat{j} + \left(\frac{9}{29} - 1\right)\hat{k}\right]$

$= \left(2\hat{i} + 3\hat{j} + \hat{k}\right) + \lambda\left[\left(\frac{-15}{29}\right)\hat{i} + \left(\frac{-10}{29}\right)\hat{j} + \left(\frac{-20}{29}\right)\hat{k}\right]$

Where λ is a parameter.

iii. The coordinates of the point P(2, 3, 1) and the equation of the plane in which the tower's bottom is located are 3x + 2y + 4z = 11.

Height of tower = $\left|\frac{3(2) + 2(3) + 4(1) - 11}{\sqrt{3^2 + 2^2 + 4^2}}\right|$

$= \left|\frac{6+6+4-11}{\sqrt{9+4+16}}\right|$

$= \frac{5}{\sqrt{29}}$

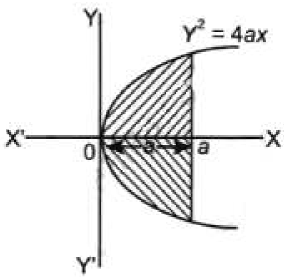
Q18.

18.1. Using integration, find the area bounded by the curve $y^2 = 4ax$ and the line $x = a$.

Solution

Given: $y^2 = 4ax$

Required area (A) = $2\int_0^a y.\,dx$



$= 2\int_0^a \sqrt{4ax}.\,dx$

$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$

$= 4\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a$

$= \frac{8}{3}\sqrt{a}\left[a^{3/2} - 0\right]$

$= \frac{8}{3}a^{\frac{1}{2}+\frac{3}{2}}$

$= \frac{8}{3}a^{\frac{4}{2}}$

$= \frac{8}{3}a^2$ sq.units.

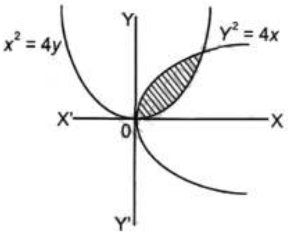
OR

18.2. Using integration, find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.

Solution

Given that the curves are $y^2 = 4x$ and $x^2 = 4y$.

Now, the graph of the provided curves is as follows:



The given equations are:

$y^2 = 4x$ ...(i)

And $x^2 = 4y$

$$y = \frac{x^2}{4} \quad ...(ii)$$

Put the value of (ii) in (i), we get

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$\Rightarrow \frac{x^4}{16} = 4x$$

$\Rightarrow x^4 = 4 \times 16x$

$\Rightarrow x^4 - 64x = 0$

$\Rightarrow x(x^3 - 64) = 0$

$\Rightarrow x = 0$ or $x = 4$

The curve is rewritten as follows:

$y^2 = 4x$

$$\Rightarrow y = \sqrt{4x} = 2\sqrt{x}$$

$\Rightarrow x^2 = 4y$

$$\Rightarrow y = \frac{x^2}{4}$$

Now, the area of the bounded region is given as:

$$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left[2 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\left(\frac{4}{3} \times 4^{3/2}\right) - \frac{4^3}{12}\right] - 0$$

$$= \left[\frac{4 \times 8}{3} - \frac{64}{12}\right]$$

$$= \frac{128 - 64}{12}$$

$$= \frac{64}{12}$$

$$= \frac{16}{3} \text{ sq.units}$$

SECTION C -15 MARKS

Q19. In subparts (i) and (ii) choose the correct options and in subparts (iii) to (v), answer the questions as instructed.

19.1. A company sells hand towels at ₹ 100 per unit. The fixed cost for the company to manufacture hand towels is ₹ 35,000 and variable cost is estimated to be 30% of total revenue. What will be the total cost function for manufacturing hand towels?

1. 35000 + 3x
2. 35000 + 30x

3. 35000 + 100x
4. 35000 + 10x

Solution

35000 + 30x

Explanation:

Here, fixed cost = ₹ 35,000

Total revenue = ₹ 100x, if x units are sold

Variable cost = $\frac{30}{100}$ × Revenue

= $\frac{30}{100} \times 100x$

= 30x

∴ Total cost function = F.C. + V.C.

= 35,000 + 30x

19.2.

If the correlation coefficient of two sets of variables (X, Y) is $\frac{-3}{4}$, which one of the following statements is true for the same set of variables?

1. Only one of the two regression lines has a negative coefficient.
2. Both regression coefficients are positive.
3. Both regression coefficients are negative.
4. One of the lines of regression is parallel to the x-axis.

Solution

Both regression coefficients are negative.

Explanation:

A negative correlation occurs when the correlation coefficient is smaller than O. Thus, if both regressions are negative, the correlation coefficient will likewise be negative. As a result, if the correlation coefficient is negative, both regression coefficients will be negative.

19.3.

If the total cost function is given by $C = x + 2x^3 - \frac{7}{2}x^2$, find the Marginal Average Cost function (MAC).

Solution

Given, $C(x) = x + 2x^3 - \frac{7}{2}x^2$

MAC = $\frac{d}{dx}(AC)$

Average cost (AC) = $\frac{C(x)}{x}$

= $\frac{x + 2x^3 - \frac{7}{2}x^2}{x}$

= $1 + 2x^2 - \frac{7}{2}x$

MAC = $\frac{d}{dx}(AC)$

= $\frac{d}{dx}\left(1 + 2x^2 - \frac{7}{2}x\right)$

= $4x - \frac{7}{2}$

19.4. The equations of two lines of regression are 4x + 3y + 7 = 0 and 3x + 4y + 8 = 0. Find the mean value of x and y.

Solution

The two regression lines are:

4x + 3y + 7 = 0 ...(i)

3x + 4y + 8 = 0 ...(ii)

We solve these equations simultaneously because the point $(\bar{x}, \bar{y})$ is on both regression lines.

Multiplying equation (i) by 4 and equation (ii) by 3 and subtracting both equations, we get

$$\begin{array}{r} 16x + 12y + 28 = 0 \\ 9x + 12y + 24 = 0 \\ - \quad - \qquad - \quad \\ \hline 7x + 4 = 0 \end{array}$$

$$x = -\frac{4}{7}$$

Putting the value of x into equation (i), we get

$$4 \times \left(-\frac{4}{7}\right) + 3y + 7 = 0$$

$$\Rightarrow -\frac{16}{7} + 7 + 3y = 0$$

$$\Rightarrow 3y = \frac{16}{7} - 7$$

$$= \frac{16 - 49}{7}$$

$$= -\frac{33}{7}$$

$$\Rightarrow y = -\frac{11}{7}$$

Hence, $\bar{x} = -\frac{4}{7}$ and $\bar{y} = -\frac{11}{7}$

19.5. The manufacturer of a pen fixes its selling price at ₹ 45 and the cost function is C(x) = 30x + 240. The manufacturer will begin to earn profit if he sells more than 16 pens. Why? Give one reason.

Solution

Revenue function, R(x) = 45x, if x piece of pens are sold.

Cost function, C(x) = 30x + 240

We will determine the breakeven points,

i.e., R(x) = C(x)

⇒ 45x = 30x + 240

⇒ 45x – 30x = 240

⇒ 15x = 240

⇒ x = $\frac{240}{15}$ = 16

As a result, if more than 16 pens are sold, profits will be made.

Q20.

20.1.

The Average Cost function associated with producing and marketing x units of an item is given by AC = $x + 5 + \frac{36}{x}$.

a. Find the Total cost function.
b. Find the range of values of x for which Average Cost is increasing.

Solution

Give, AC(x) = $x + 5 + \frac{36}{x}$

a. T.C(x) = x × AC

= $x\left(x + 5 + \frac{36}{x}\right)$

= x^2 + 5x + 36

b. Given, AC = $x + 5 + \frac{36}{x}$

$$\frac{d}{dx}(AC) = 1 - \frac{36}{x^2}$$

AC increases, when $\frac{d}{dx}(AC) > 0$

$\Rightarrow 1 - \frac{36}{x^2} > 0$

$\Rightarrow 1 > \frac{36}{x^2}$

$\Rightarrow x^2 > 36$

$\Rightarrow x^2 - 36 > 0$

$\Rightarrow (x - 6)(x + 6) > 0$

$\Rightarrow x > 6$ or $x < -6$

As x is positive, AC increases when $x > 6$

OR

20.2. A monopolist's demand function is $x = 60 - \frac{P}{5}$. At what level of output will marginal revenue be zero?

Solution

Demand function, $x = 60 - \frac{p}{5}$ ...(i)

Total revenue function, R(x) = px

= 5(60 – x)x ...[From (i)]

= 5x(60 – x)

= $300x - 5x^2$

Then, MR = $\frac{d}{dx}(R(x))$

= $\frac{d}{dx}(300x - 5x^2)$

= 300 – 10x

Put MR = 0

⇒ 300 – 10x = 0

⇒ 10x = 300

⇒ x = 30

Hence, the value of output is 30.

Q21.

21.1.

A monopolist's demand function is $x = 60 - \frac{p}{5}$. At what level of output will marginal revenue be zero?

Solution

Demand function, $x = 60 - \frac{p}{5}$ ...(i)

Total revenue function, R(x) = px

= 5(60 – x)x ...[From (i)]

= 5x(60 – x)

= 300x – $5x^2$

Then, MR = $\frac{d}{dx}(R(x))$

= $\frac{d}{dx}(300x - 5x^2)$

= 300 – 10x

Put MR = 0

⇒ 300 – 10x = 0

⇒ 10x = 300

⇒ x = 30

Hence, the value of output is 30.

21.2.

For 50 students of a class, the regression equation of marks in statistics (X) on the marks in accountancy (Y) is 3y – 5x + 180 = 0. The mean marks in accountancy is 44 and the variance of marks in statistics is $\left(\frac{9}{16}\right)^{th}$ of the variance of marks in accountancy. Find the mean marks in statistics and the correlation coefficient between marks in the two subjects.

Solution

Given n = 50

Regression line of x on y is

3y – 5x + 180 = 0

⇒ 5x = $\frac{3y+180}{5}$

∴ x = $\frac{3}{5}y+\frac{180}{5}$

$\therefore b_{xy} =$ coefficient of y $= \frac{3}{5}$

Variance of marks in statistics = $\frac{9}{16}$

Variance of marks in accountancy.

i.e. V(x) = $\frac{9}{16}$V(y)

$\Rightarrow \frac{V_x}{V_y} = \frac{9}{16}$

Taking square roots,

$\frac{\sigma_x}{\sigma_y} = \frac{3}{4}$

We have, $b_{xy} = r.\frac{\sigma_x}{\sigma_y}$

$\Rightarrow \frac{3}{5} = r \times \frac{3}{4}$

$\Rightarrow r = \frac{4}{5} = 0.8$

Given $\overline{y}$ = 44

∴ Substituting y = 44 is 3y – 5x + 180 = 0

⇒ 3(44) – 5x + 18 = 0

⇒ 132 – 5x + 180 = 0

⇒ 5x = 132 + 180 = 312

⇒ x = $\frac{312}{5}$ = 62.4

∴ $\overline{x}$ = 62.4

∴ Mean marks in statistics $\overline{x}$ = 62.4

Q22. Aman has ₹ 1500 to purchase rice and wheat for his grocery shop. Each sack of rice and wheat costs ₹ 180 and Rupee ₹ 120 respectively. He can store a maximum number of 10 bags in his shop. He will earn a profit of ₹ 11 per bag of rice and ₹ 9 per bag of wheat.

i. Formulate a Linear Programming Problem to maximise Aman's profit.
ii. Calculate the maximum profit.

Solution

i. Let x be the number of rice sacks purchased and y be the number of wheat sacks purchased.

Cost of each sack of rice = ₹ 180

Cost of each sach of wheat = ₹ 120

Profit earned on each sack of rice = ₹ 11

Profit earned on each sack of wheat = ₹ 9

As a result, the problem can be expressed as LPP as follows:

Maximum P = 11x + 9y,

Subject to the constraints

180x + 120y ≤ 1500

or 3x + 2y ≤ 25 ...(i)

x + y ≤ 10 ...(ii)

x, y > 0

ii. Draw the lines 3x + 2y = 25 and x + y = 10 and shade the area bounded by the given inequations. The shaded region represents a bounded feasible region. Lines 3x + 2y = 25 and x + y = 10 intersect at (5, 5).

For equation (i)

3x + 2y = 25

x	0	$\frac{25}{2}$
y	$\frac{25}{3}$	0

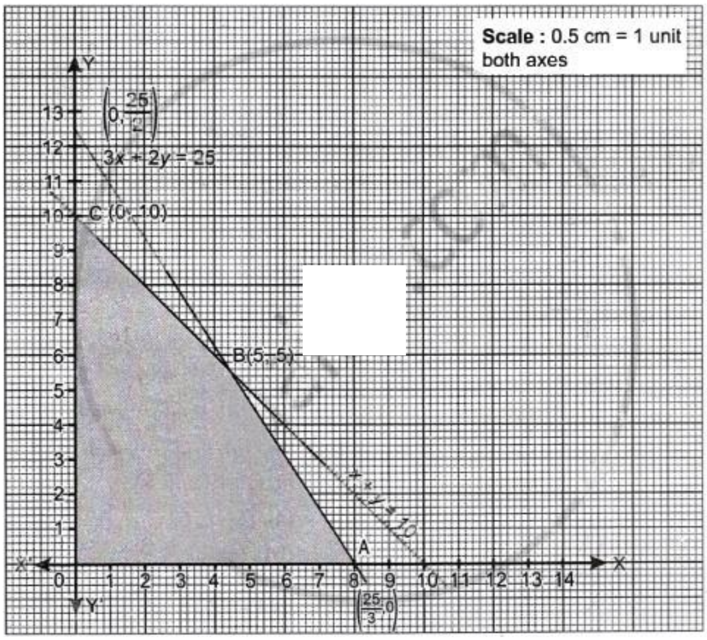
Points: $\left(0, \frac{25}{2}\right), \left(\frac{25}{3}, 0\right)$

For equation (ii)

x + y = 10

x	0	10
y	10	0

Points: (0, 10), (10, 0)



The four corner points of the feasible region OABC are O(0, 0), $A\left(\frac{25}{3}, 0\right)$, B(5, 5), C(0, 10)

At each corner point, we calculate Z = 11x + 9y.

Corner points	**Value of objective function Z = 11x + 9y**
O(0, 0)	Z = 11 × 0 + 9 × 0 = 0
$A\left(\frac{25}{3}, 0\right)$	$Z = 11 \times \frac{25}{3} + 9 \times 0 = \frac{275}{3}$
B(5, 5)	Z = 11 × 5 + 9 × 5 = 100
C(0, 10)	Z = 11 × 0 + 9 × 10 = 90

P is found to be greatest at B(5, 5), with a maximum value of P = 100.

Hence, Aman earns the greatest profit of ₹ 100 when purchasing 5 sacks of rice and 5 sacks of wheat.