

Scalar Triple Product

Q.1. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, show that : $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) \times (\vec{c} + \vec{a}) = 2[\vec{a} \vec{b} \vec{c}]$

Solution : 1

$$\begin{aligned} \text{L.H.S.} &= (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) \times (\vec{c} + \vec{a}) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) \\ &= \vec{a} \cdot \vec{b} \times \vec{c} + \vec{a} \cdot \vec{b} \times \vec{a} + \vec{a} \cdot \vec{c} \times \vec{a} + \vec{b} \cdot \vec{b} \times \vec{c} \\ &\quad + \vec{b} \cdot \vec{b} \times \vec{a} + \vec{b} \cdot \vec{c} \times \vec{a} \\ &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{c}] \quad [\text{As, } \vec{a} \cdot \vec{b} \times \vec{a} = 0, \vec{b} \cdot \vec{b} \times \vec{a} = 0, \\ &= 2[\vec{a} \vec{b} \vec{c}] \quad \text{[Proved.] \& } \vec{b} \cdot \vec{b} \times \vec{c} = 0 \& \vec{a} \cdot \vec{c} \times \vec{a} = 0.] \end{aligned}$$

Q.2. The vectors $\vec{i} + 3\vec{j}$, $5\vec{k}$ and $\lambda\vec{i} - \vec{j}$ are coplanar. Find the value of λ .

Solution : 2

Vectors $\vec{i} + 3\vec{j}$, $5\vec{k}$ and $\lambda\vec{i} - \vec{j}$ are coplanar, [When \vec{a}, \vec{b} and \vec{c} are coplanar,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0]$$

$$\text{Therefore, } \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 5 \\ \lambda & -1 & 0 \end{vmatrix} = 0$$

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$$\text{Or, } 1\{(0 - (-5))\} + \lambda(15) = 0$$

$$\text{Or, } 15\lambda = -5$$

$$\text{Or, } \lambda = -1/3.$$

Q.3. Prove that $a \rightarrow \cdot (b \rightarrow + c \rightarrow) \times (a \rightarrow + 2b \rightarrow + 3c \rightarrow) = [a \rightarrow b \rightarrow c \rightarrow]$

Solution : 3

$$\begin{aligned}
 \text{L.H.S.} &= a \rightarrow \cdot (b \rightarrow + c \rightarrow) \times (a \rightarrow + 2b \rightarrow + 3c \rightarrow) \\
 &= a \rightarrow \cdot [b \rightarrow \times a \rightarrow + 2b \rightarrow \times b \rightarrow + 3b \rightarrow \times c \rightarrow + c \rightarrow \times a \rightarrow + 2c \rightarrow \times b \rightarrow + 3c \rightarrow \times c \rightarrow] \\
 &= a \rightarrow \cdot [- a \rightarrow \times b \rightarrow + 0 + 3b \rightarrow \times c \rightarrow + c \rightarrow \times a \rightarrow - 2b \rightarrow \times c \rightarrow] \\
 &[\text{As, } b \rightarrow \times b \rightarrow = c \rightarrow \times c \rightarrow = 0] \\
 &= a \rightarrow \cdot [- a \rightarrow \times b \rightarrow + b \rightarrow \times c \rightarrow + c \rightarrow \times a \rightarrow] \\
 &= - a \rightarrow \cdot (a \rightarrow \times b \rightarrow) + a \rightarrow \cdot (b \rightarrow \times c \rightarrow) + a \rightarrow \cdot (b \rightarrow \times a \rightarrow) \\
 &= 0 + a \rightarrow \cdot (b \rightarrow \times c \rightarrow) + 0 [\text{As, } a \rightarrow \cdot (a \rightarrow \times b \rightarrow) = (a \rightarrow \times a \rightarrow) \cdot b \rightarrow = 0] \\
 &= a \rightarrow \cdot (b \rightarrow \times c \rightarrow) = \text{R.H.S. [Proved.]}
 \end{aligned}$$

Q.4. Find the volume of the parallelepiped whose three co-terminus edges are represented by the vectors : $i + j + k$, $i - j + k$ and $i + 2j - k$.

Solution : 4

$$\begin{aligned}
 & \qquad \qquad \qquad | 1 \ 1 \ 1 | \\
 \text{Volume of parallelepiped} &= | 1 \ -1 \ 1 | \\
 & \qquad \qquad \qquad | 1 \ 2 \ -1 | \\
 &= 1(1 - 2) - 1(-1 - 1) + 1(2 + 1) \\
 &= -1 + 2 + 3 \\
 &= 4 \text{ cubic units.}
 \end{aligned}$$

Q.5. Find the volume of the parallelepiped whose edges (co-terminus) are represented by the vectors : $a \rightarrow = 2i - 3j + 4k$, $b \rightarrow = i + 2j - k$, $c \rightarrow = 3i - j + 2k$.

Solution : 5

We have, $\vec{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\vec{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\vec{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Volume, $V = (\vec{a} \times \vec{b}) \cdot \vec{c} = [(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})] \cdot (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 3(3 - 8) - (-1)(-2 - 4) + 2(4 + 3)$$

$$= -15 - 6 + 14 = -7.$$

Hence, $V = |-7| = 7$ cubic units.