Integration by Substitution

Q.1. Solve : ∫1/[x $\cos^2$ (1 + log x)] dx.

Solution : 1

We have ∫1/[x cos ^{2}(1 + log x)] dx ,

Put 1 + log x = t

Then 1/x dx = dt

Therefore, ∫(1/x){1/(1 + log x)}dx

= ∫ dt/$\cos^2$t = ∫$\sec^2$t dt

= tan t + c

= tan (1 + log x) + c .

Q.2. Evaluate : ∫x^2(e^x3)cos(2e^x^3)dx.

Solution : 2

We have ∫x^2(e^x^3)cos(2e^x^3)dx ,

Put 2e^x^3 = t then 2e^x^3 × 3x^2 = dt/dx

Or, dx = dt/[2e^x^3×3x^2]

Then I = ∫ x^2(e^x^3)cos(t)[dt/(2e^x^3×$^3x^2$)]

= 1/6∫cost dt = 1/6 sin t + c = 1/6 sin(2e^x3) + c.

Q.3. Evaluate : ∫ cos x/[sin x + √(sin x)] dx .

Solution : 3

I = ∫ cos x/[sin x + √(sin x)] dx , put sin x = t^{2} , cos x dx = 2tdx .

Therefore , I = 2 ∫ tdt/(t^{2} + t) = 2 ∫tdx/[t(t + 1)]

= 2∫dt/(t + 1) = 2 log│t + 1│ + c

= 2 log │√(sin x) + 1│+ c.

Q.4. Integrate : ∫(1 + $\tan^{2}$x)/[√(1 – $\tan^{2}$x)]dx .

Solution : 4

Let I = ∫(1 + $\tan^{2}$x)/[√(1 – $\tan^{2}$x)]dx

Putting tan x = t , then $\sec^{2}$xdx = dt => (1 + $\tan^{2}$x)dx = dt

Therefore , I = ∫dt/√(1 – t^{2})

= sin −1 t

= sin −1 (tan x) + c.

Q.5. Evaluate : ∫[x^{2}/√(x + 1)]dx.

Solution : 5

Let I = ∫[x^{2}/√(x + 1)] dx and put x = 1 = t^{2} => dx = 2tdt

Therefore, I = ∫[{(t^{2} – 1)^{2}}/t].2tdt

= 2∫(t^{4} – 2t^{2} + 1).dt

= 2[t5/5 – 2×t^{3}/3 + t] + c

$= 2t[(1/5)\ t^4 - (2/3)\ t^2 + 1] + c$

$= 2\sqrt{(x + 1)}[(1/5)(x + 1)^2 - (2/3)(x + 1) + 1] + c.$

Q.6. Evaluate : $\int\sqrt{(1 + \sin 1/2\ x)}\ dx$

Solution : 6

Let $I = \int\sqrt{(1 + \sin 1/2\ x)}dx$ and put $x = 2t \Rightarrow dx = 2dt$

Therefore, $I = \int\sqrt{(1 + \sin t)}.\ 2dt$

$= 2\int\sqrt{(\sin^2 t/2 + \cos^2 t/2 + 2 \sin t/2.\cos t/2)}dt$

$= 2\int\sqrt{\{(\sin t/2 + \cos t/2)2\}}dt$

$= 2\int(\sin t/2 + \cos t/2).dt$

$= 2\ [-2 \cos t/2 + 2 \sin t/2] + c$

$= 4[\sin x/4 - \cos x/4] + c.$

Q.7. Evaluate : $\int[\{\sin^2(\log x)\}/x]dx$.

Solution : 7

Let $I = \int[\{\sin^2(\log x)\}/x].dx$ and put $\log x = t \Rightarrow (1/x)dx = dt$

Therefore, $I = \int\sin^2 t.dt = 1/2\ \int(1 - \cos 2t).dt$

$= 1/2\ [t - (1/2) \sin 2t] + c$

$= 1/2\ [\log x - (1/2) \sin (2 \log x)] + c.$

Q.8. Evaluate : $\int[x^2/\{(a + bx)^2\}].dx$.

Solution : 8

Let I = $\int[x^2/\{(a + bx)^2\}].dx$ and put a + bx = t => bdx = dt => dx = dt/b

Therefore, I = $1/b^2\int[\{(bx)^2\}/\{(a + bx)^2].dx$

= $1/b^2\int[\{(t - a)^2\}/\{t^2\}.(dt/b)]$

= $1/b^3\int[(t^2 - 2at + a^2)/(t^2)].dt$

= $1/b^3\int[1 - (2a/t) + a^2t-2].dt$

= $1/b^3[t - 2a \log t - a^2/t] + c$

= $1/b^3[(a + bx) - 2a \log|(a + bx)| - a^2/(a + bx)] + c.$

Q.9. Evaluate : $\int\sqrt{(2 + \sin 3x)}.\cos 3x .dx.$

Solution : 9

Let I = $\int\sqrt{(2 + \sin 3x)}.\cos 3x .dx$ and put 2 + sin 3x = t => (1/3)cos 3x dx = dt

Therefore, I = $\int\sqrt{(t)}.dt/3 = (1/3) \int\sqrt{t} .dt$

= $(1/3)(2/3)(t)^{3/2} + c$

= $(2/9)(2 + \sin 3x)^{3/2} + c.$

Q.10. Evaluate : $\int[x^{1/2}/(x^{1/2} - x^{1/3})].dx.$

Solution : 10

Let I = $\int[x^{1/2}/(x^{1/2} - x^{1/3})].dx$ and put $x = t^6$ => $dx = 6t^5dt$

[LCM of 2 &3 is 6]

Therefore, I = $\int[t^3/(t^3 - t^2)].6t^5dt$

= 6 $\int[t^8/t^2(t - 1)]dt = 6\int[t^6/(t - 1)]dt$

= $6\int[t^5 + t^4 + t^3 + t^2 + t + 1 + 1/(t - 1)]dt$ [dividing t^6 by t – 1]

= $6[(1/6)t^6 + (1/5)t^5 + (1/4)t^4 + (1/3)t^3 + (1/2)t^2 + t + \log|t - 1|] + c$

= $6[(1/6)x + (1/5)x^{5/6} + (1/4)x^{4/6} + (1/3)x^{3/6} + (1/2)x^{2/6} + x^{1/6} + \log|x1/6 - 1|] + c$.

Q.11. Evaluate : $\int[(10x^9 + 10^x \log_e 10)/(10^x + x^{10})]dx$.

Solution : 11

Let I = $\int(10x^9 + 10^x \log_e 10)/(10^x + x^{10})dx$

Put $10^x + x^{10} = t$, then $(10x^9 + 10^x \log_e 10)dx = dt$

Therefore, I = $\int dt/t$

= loge|t|+ c = $\log_e|10^x + x^{10}| + c$.

Q.12. Evaluate : $\int[1/(1 + \cot x)].dx$.

Solution : 12

Let I = $\int[1/(1 + \cot x)]dx = \int[\sin x\, dx/(\sin x + \cos x)]dx$

= $\int[\{(\sin x + \cos x) - (\cos x - \sin x)\}/\{2(\sin x + \cos x)\}]dx$

= $(1/2)\int dx - (1/2)\int[(\cos x - \sin x)/(\cos x + \sin x)]dx$ Put cos x + sin x

= t => $(-\sin x + \cos x)dx = dt$

Therefore, I = $(1/2)x - (1/2)\int dt/t$

$= (1/2) - (1/2) \log |t| + c$

$= (1/2)x - (1/2) \log |\cos x + \sin x| + c.$

Q.13. Evaluate : $\int[\sin^2 x/(a^2 + b^2 \sin^2 x)].dx$

Solution : 13

Let $I = \int[\sin 2x/(a^2 + b^2 \sin^2 x)].dx.$

Put $a^2 + b^2\sin^2 x = t \Rightarrow b^2.(2\sin x \cos x)dx = dt \Rightarrow \sin {}^2xdx = dt/b^2$

Therefore, $I = 1/b^2\int dt/t = 1/b^2 \log |t| + c$

$= 1/b^2 \log|(a^2 + b^2 \sin^2 x)| + c.$

Q.14. Evaluate : $\int a/(b + ce^x).dx.$

Solution : 14

Let $I = \int a/(b + ce^x).dx = \int ae^{-x}/(be^{-x} + c).dx$

Put $be^{-x} + c = t \Rightarrow - be^{-x} dx = dt$

Therefore, $I = (-a/b)\int dt/t = - a/b \log |t| + k - a/b \log|(be^{-x} + c| + k$

$= - a/b \log|b.(1/e^x) + c| + k$

$= - a/b \log [|b + ce^x| - \log |e^x| + k$

$= - a/b[\log|b + ce^x | - x] + k.$

Q.15. Evaluate : $\int[1/(e^x + e^{-x})].dx$

Solution : 15

Let I = $\int[1/(e^{x} + e^{-x})].dx = \int[e^{x}/(e^{2x} + 1)].dx$

Put $e^{x} = t \Rightarrow e^{x}dx = dt$

Therefore, I = $\int dt/(t^{2} + 1) = \tan^{-1}(t) + c = \tan^{-1}(ex) + c$.

Q.16. Evaluate : $\int[e^{2x}/(e^{x} + 1)].dx$.

Solution : 16

Let I = $\int[e^{2x}/(e^{x} + 1)].dx = \int[(e^{x}.e^{x})/(e^{x} + 1)].dx$

Put $ex = t \Rightarrow e^{x}dx = dt$

Therefore, I = $\int[t/(t + 1)]dt = \int[\{(t + 1) - 1\}/(t + 1)]dt$

$= \int[1 - \{1/(t + 1)\}]dt = t - \log|t + 1| + c$

$= e^{x} - \log|e^{x} + 1| + c$.

Q.17. Evaluate : $\int[(e^{x} + e^{-x})/(e^{x} - e^{-x})].dx$.

Solution : 17

Let I = $\int [(e^{x} + e^{-x})/(e^{x} - e^{-x})].dx$

Put $ex - e^{-x} = t \Rightarrow (e^{x} + e^{-x})dx = dt$.

Therefore, I = $\int(1/t).dt = \log|t| + c = \log|e^{x} - e^{-x}| + c$.

Q.18. Evaluate : $\int\{\cot(\log x)/x\}.dx$.

Solution : 18

Let I = ∫[cot(log x)].dx. And put log x = z => (1/x)dx = dz

Therefore, I = ∫cotz.dz = log |sin z| + c

= log|sin (log x)| + c.

Q.19. Evaluate : ∫[cos (x – α)/cos x].dx.

Solution : 19

Let I = ∫[cos (x – α)/cos x].dx

= ∫[(cos x cos α+ sin x sin α)/cos x]dx

= ∫[cos α + tan x sin α].dx

= cos α ∫dx + sin α ∫tan x dx

= (cos α) x + sin α log |sec x| + c.

Q.20. Evaluate : ∫dx/√(1 + sin x).

Solution : 20

Let I = ∫dx/√(1 + sin x)

= ∫dx/√{1 – cos (π/2 + x)}

= ∫dx/√{2 sin2 (π/4 + x/2)}

= (1/√2)∫cosec (π/4 + x/2).dx

Put π/4 + x/2 = t => dx = 2dt

Therefore, I = (2/√2)∫cosec t dt

= √2 log |tan (t/2)| + c

= √2 log |tan (π/8 + x/4)| + c.