Integration by Partial Fraction

Q.1. Evaluate : $\int[(x - 1)/\{(x - 3)(x - 2)^2\}]\, dx$.

Solution : 1

We are given $\int[(x - 1)/\{(x - 3)(x - 2)2\}]dx$.

Now, let $(x - 1)/\{(x - 3)(x - 2)2\} = A/(x - 3) + B/(x - 2) + C/(x - 2)^2$

Then $x - 1 = A(x - 2)^2 + B(x - 3)(x - 2) + C(x - 3)$

Put $x = 2$, $1 = C(2 - 3) = - C \Rightarrow C = -1$.

Put $x = 3$, $2 = A(3 - 2)^2 = A \Rightarrow A = 2$.

Put $x = 0$, $-1 = A(0 - 2)^2 + B(0 - 3)(0 - 2) + C(0 - 3) = 4A + 6B - 3C$

$= 4\times2 + 6B - 3\times(-1) = 6B + 11 \Rightarrow 6B = -12 \Rightarrow B = - 2$.

Putting the values of A,B and C we get

$\int[(x - 1)/\{(x - 3)(x - 2)2\}]\, dx = \int[2/(x - 3)]\, dx - \int[2/(x - 2)]\, dx - \int[1/(x - 2)2]\, dx$

$= 2 \log (x - 3) - 2\log(x - 2) + 1/(x - 2) + c$.

Q.2. Evaluate : $\int dx/(2x^2 + x - 1)$.

Solution : 2

Let $I = \int dx/(2x^2 + x - 1)$;

Let $1/(2x^2 + x - 1) = 1/[(2x - 1)(x + 1)]$

$= 1/[(2x - 1)(1/2 + 1)] + 1/[\{2 (-1) -1\}(x + 1)]$

= (2/3)/(2x − 1) − (1/3)/(x + 1) .

Therefore, I = (2/3)∫dx/(2x − 1) − (1/3)∫dx/(x + 1)

= (2/3)[(1/2) log |2x − 1|− (1/3) log |x + 1| + c

= (1/3) log|(2x − 1)/(x + 1)|+ c.

Q.3. Evaluate : ∫[x/{(x^2 − a^2)(x^2 − b^2)}].dx.

Solution : 3

Let I = ∫[x/{(x^2 − a^2)(x^2 − b^2)}].dx

[Put x^2 = t then 2xdx = dt => xdx = (1/2)dt]

Therefore, I = (1/2)∫dt/[(t − a2)(t − b2)] ;

Now 1/[(t − a^2)(t − b^2)] = 1/[(t − a^2)(a^2 − b^2)] − 1/[(t − b^2)(a^2 − b^2)]

Therefore, I = 1/{2(a^2 − b^2)}∫[1/(t − a^2) − 1/(t − b^2)].dt

= 1/{2(a^2 − b^2)}[log |t − a^2|− log|t − b^2|] + c

= 1/{2(a^2 − b^2)}[log|x^2 − a^2|− log|x^2 − b^2|+ c

= 1/{2(a^2 − b^2)}log|(x^2 − a^2)/(x^2 − b^2)|+ c.

Q.4. Evaluate : ∫[cos x/{(1 + sin x)(2 + sin x)}].dx.

Solution : 4

Let I = ∫[cos x/{(1 + sin x)(2 + sin x)}].dx

[Put sin x = t then cos x dx = dt]

Then I = ∫dt/{(1 + t)(2 + t)}

$= \int[1/\{(1 + t)(2 - 1)\} + 1/\{(1 - 2)(2 + t)\}].dt$

$= \int[1/(1 + t) - 1/(2 + t)].dt$

$= \log|1 + t| - \log|2 + t|$

$= \log|(1 + t)/(2 + t)| + c$

$= \log|(1 + \sin x)/(2 + \sin x)| + c.$

Q.5. Evaluate : $\int dx/(\sin x + \sin 2x)$.

Solution : 5

Let $I = \int dx/(\sin x + \sin 2x)$

$= \int dx/(\sin x + 2 \sin x \cos x)$

$= \int dx/\{\sin x (1 + 2\cos x)\}$

$= \int[\sin x/\{\sin2 x(1 + 2\cos x)\}].dx$

$= \int[\sin x/\{(1 - \cos2 x)(1 + 2\cos x)\}].dx$

[Put $\cos x = t$ then $- \sin x\, dx = dt$]

Then $I = - \int dt/\{(1 - t2)(1 + 2t)\}$

$= - \int dt/\{(1 + t)(1 - t)(1 + 2t)\}$;

Now $1/\{(1 + t)(1 - t)(1 + 2t)\}$

$= 1/[(1 + t)\{1 - (-1)\}\{1 + 2(-1)] + 1/[(1 + 1)(1 - t)\{1 + 2(1)\}] + 1/[(1 - 1/2)\{1 - (-1/2\} (1 + 2t)]$

$= (-1/2)\{1/(1 + t)\} + (1/6)\{1/(1 - t)\} + (4/3)\{1/(1 + 2t)\}$

Therefore, $I = (1/2)\int dt/(1 + t) - (1/6)\int dt/(1 - t) - (4/3)\int dt/(1 + 2t)$

$= (1/2)\log|1 + t| + (1/6) \log|1 - t| - (4/3)(1/2) \log|1 + 2t| + c$

$= (1/2) \log|1 + \cos x| + (1/6) \log|1 - \cos x| - (2/3) \log|1 + 2 \cos x| + c.$

Q.6. Evaluate : $\int dx/[x(x^5 + 1)]$.

Solution : 6

Let $I = \int dx/[x(x^{5} + 1)]$

$= \int[x^{4}/\{x^{5} (x^{5} + 1)\}].dx$ [Multiplying Nr & Dr by x4]

[Put $x^{5} = t$ then $x^{4}dx = (1/5)dt$]

Therefore, $I = (1/5)\int dt/[t(t + 1)]$

$= (1/5)\int[1/t - 1/(t + 1)].dt$

$= (1/5) [\log|t| - \log|t + 1|] + c$

$= (1/5) \log|t/(t + 1)| + c$

$= (1/5) \log |x^{5}/(x^{5} + 1)| + c.$