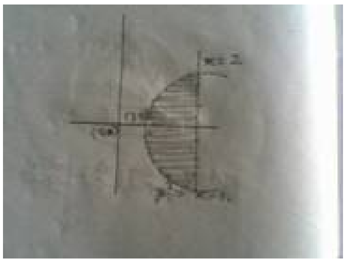
Area and Volume of Curve

Q.1. Draw a rough sketch of the curve $y^2 + 1 = x$, $x \le 2$. Find the area enclosed by the curve and the line $x = 2$.

Solution : 1



Fig

We have , $y^{2} + 1 = x$, $x \le 2$,

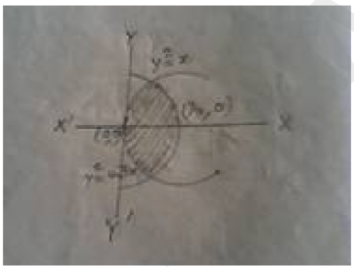
Or, $y^{2} = x - 1$.

Required area = $2\ (1, 2) \int \sqrt{(x - 1)}\ dx$

= $\{2[(x - 1)](1, 2)\}/(3/2) = 2 \times (2/3)\ [(1)3/2] = 4/3$ sq. units.

Q.2. Find the area enclosed by the curve $y^2 = x$ and $y^2 = 4 - 3x$.

Solution : 2



Fig

We have , $y^{2} = x$ and $y^{2} = 4 - 3x$,

Solving these two equations simultaneously ,

$x = 4 - 3x \Rightarrow 4x = 4 \Rightarrow x = 1$ and $y = \pm 1$.

The points are $(0, 0)$, $(4/3, 0)$, $(1, 1)$ and $(1 , - 1)$.

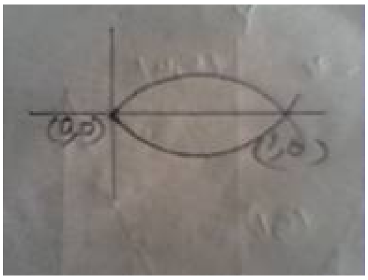
The required area = $2[01\int \sqrt{x}\ dx + 1^{4/3}\int \sqrt{(4 - 3x)}\ dx\ ]$

$= 2[\{(2\ x^{3/2})/3\}01 + \{2(4 - 3x)^{3/2}/(-\ 3 \times 3\ )\}1^{4/3}]$

$= 2[2/3 + 2/9]\ 2[(6 + 2)/9] = 16/9$ sq. units.

Q.3. The region bounded by the curve $y^2 = x(x - 1)^2$, the x –axis and the lines x = 1, x = 0 , is rotated through four right angles about the x-axis. Calculate the volume of the solid of revolution so formed.

Solution : 3



Fig

Curve $y^2 = x(x - 1)^2$

Volume $= \pi 0\int 1\ y^2\ dx = \pi 0\int 1\ x(x - 1)\ dx$

$= \pi 0\int 1\ (x^3 - 2x^2 + x)\ dx$

$= \pi[x4/4 - 2x3/3 + x^2/2]0^1$

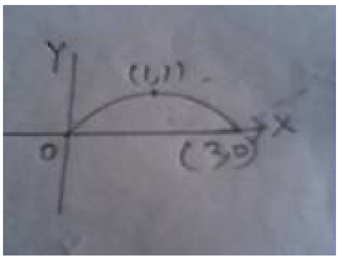
$= \pi[1/4 - 2/3 + 1/2]$

$= \pi[(3 - 8 + 6)/12]$

$= \pi/12$ cubic unit.

Q.4. Calculate the area bounded by the curve y = x(2 − x) and the line x = 0 , y = 0 , x = 2. This area is rotated through four right angles about the x-axis. Calculate the volume of the solid so formed.

Solution : 4



Fig

Area = $0\int 2x(2 - x)dx = 0\int^{2}(2x - x^{2})dx$

$= [2x^{2}/2 - x^{3}/3]0^{2}$

$= 4 - 8/3 = 4/3$ sq. units.

Volume = $\pi 0\int 2y^{2}dx$

$= \pi 0\int 2x^{2}(2 - x)^{2}\,dx$

$= \pi 0\int 2(4x^{2} - 4x^{3} + x^{4})\,dx$

$= \pi[4x^{3}/3 - 4x^{4}/4 + x^{5}/5]0^{2}$

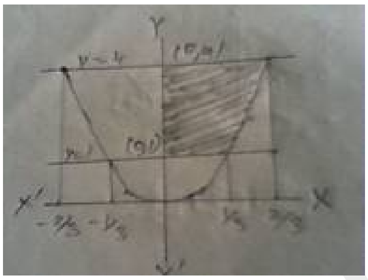
$= \pi[32/3 - 16/1 + 32/5]$

$= \pi[(160 - 240 + 96)/15]$

$= 16/15\pi$ cubic units.

Q.5. Sketch and shade the area of the region lying in the first quadrant and bound by y = 9x2, x = 0, y = 1 and y = 4.Find the area of the shaded region.

Solution : 5



Fig

We have $y = 9x^{2}$,

at x = 0 , $y = 9(0)^{2} = 0$ and at y = 4, $x = \sqrt{(4/9)} = \pm 2/3$, at y = 1, $x = \sqrt{(1/9)} = \pm 1/3$.

Atrea of shaded region = $1\int^{4}\{(y^{1/2})/3\}.dy$ [As, $\int x.dy = \int\sqrt{(y/9)}.dy = \int(\sqrt{y})/3\,.dy$]

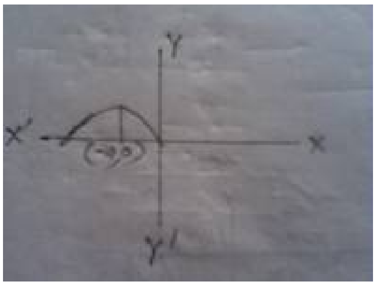
$= 1/3[(y^{3/2}(3/2)]$

$= 2/9[4^{3/2} - 1^{3/2}]$

$= 2/9[8 - 1] = 14/9 = 1.55$ sq.unit.

Q.6. Show that the area included between the x-axis and the curve $a^{2}y = x^{2}2(x + a)$ is $a^{2}/12$.

Solution : 6



Fig

We have $a^{2}y = x^{2}(x + a)$

When $y = 0$, $x = -a$.

Therefore, Area = $-a\int^{0} ydx = -a\int 0[(x^{3} + ax^{2})/a^{2}]dx$

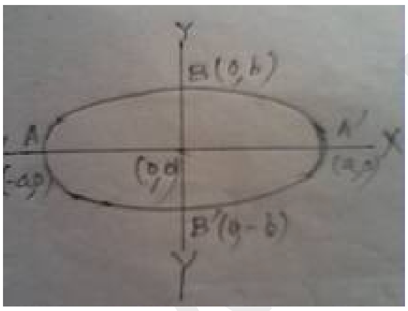
$= 1/a^{2}[x^{4}/4 + ax^{3}/3]-a^{0}$

$= 1/a^{2}[(0 + 0) - (a^{4}/4 - a^{4}/3)$

$= a^{2}/12$ units .

Q.7. Find the volume of the solid obtained by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the axis of x.

Solution : 7



Fig

If the curve $y = f(x)$ revolves round x-axis, then the volume generated = $\int \pi y^{2}.dx$

Here we have $x^{2}/a^{2} + y^{2}/b^{2} = 1$

Or, $y^{2}/b^{2} = 1 - x^{2}/a^{2}$

Or, $y^{2} = b^{2}/a^{2}.(a^{2} - x^{2})$

Therefore, Volume of the solid = $\int \pi.b^{2}/a^{2}.(a^{2} - x^{2}).dx$

$= (\pi \times b^{2})/a^{2} -a\int a(a^{2} - x^{2}).dx$

$= (\pi b^2)/a^2\ [a^2x - x^3/3]\ a{-}a$

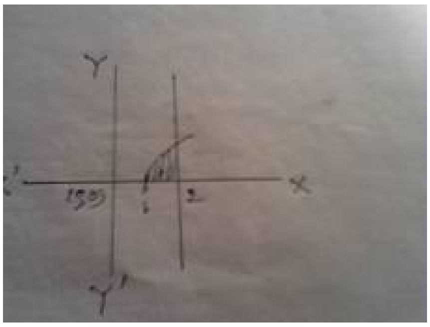
$= (\pi b^2)/a^2[a^3 - a^3/3 + a^3 - a^3/3]$

$= \{(\pi b2)/a^2\}\times(4/3)a^3$

$= (4/3)\pi b^2a$ cu. unit.

Q.8. Calculate the area of the figure bounded by the curve y = log x, the straight line x = 2 and the x-axis.

Solution : 8



Fig

We have, y = log x, when x = 1, y = log 1 = 0 , when x = 2, y = log 2 = 0.6931.

Area of shaded region $= 1\int^2 ydx = 1\int^2\log x\ dx$

$= 1\int^2 1.\ \log x\ dx = x \log x - 1\int^2 (1)\ dx$

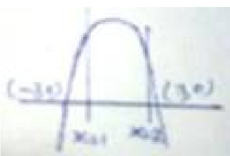
$= [x \log x - x\ ]1^2$

$= [2\log\ ^2 - 2] - [\log 1 - 1\ ]$

$= 2 \log\ ^2 - 1.$

Q.9. Draw a rough sketch of the curve $x^2 + y = 9$ and find the area enclosed by the curve, the x-axis and the line x + 1 = 0 and x − 2 = 0.

Solution : 9



Fig

Required area = $-1\int^{2}(9 - x^{2})dx = [9x - x^{3}/3]-12$

$= [(18 - 8/3) - (-9 + 1/3)]$

= [27 − 3] = 24 sq. units.

Q.10. Find the area of the figure bounded by the graphs of the functions $y = x^2$ and $y = 2x - x^2$.

Solution : 10

We have, $y = x^{2}$ and $y = 2x - x^{2}$;

Therefore, $x^{2} = 2x - x^{2}$

Or, $2x^{2} - 2x = 0$

Or, $2x(x - 1) = 0$

Therefore , x = 0 ; 1.

When x = 0 , y = 0 and when x = 1 , $y = (1)^{2} = 1$.

Now , y = x2 represents a parabola and its vertex is (0, 0).

And $y = 2x - x^{2} \Rightarrow y - 1 = 2x - x^{2} - 1 = 0 - (x^{2} - 2x + 1) = -(x - 1)^{2}$.

This represents a parabola with vertex (1, 1).

Area of the common region = $0\int^{1}[(2x - x^{2}) - x^{2}]\,dx = 0\int^{1}(2x - x^{2})dx - 0\int 1x^{2}dx$

$= [x^{2} - x^{3}/3]01 - [x^{3}/3]01$

$= (1 - 1/3) - (1/3)$

= 2/3 − 1/3 = 1/3 sq. units.