De Moiver's Theorem

Q.1. Prove that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$ by using De Moiver's Theorem.

Solution : 1

Let r cos θ = 1 and r sin θ = √3

Then tan θ = √3 => θ = π/3 and $r = \sqrt{(\sin^2 \theta + \cos^2 \theta)} = \sqrt{(1 + 3)} = 2$.

Therefore, L.H.S. = $(1+ i\sqrt{3})^8 + (1 - i\sqrt{3})^8$

$= [r(\cos \theta + i\sin \theta)]^8 + [r(\cos \theta - i\sin \theta)]^8$

$= 2^8[(\cos \pi/3 + i\sin\pi/3)8 + (\cos \pi/3 - i\sin \pi/3)8]$

$= 2^8[\cos 8/3\ i\sin 8\pi/3 + \cos 8\pi/3 - i\sin 8\pi/3]$

$= 2^8.2\cos 8\pi/3$

$= 2^8.2 \cos (3\pi - \pi/3)$

$= -2^8.2 \cos \pi/3$

$= -2^8.2 \times(1/2)$

$= -2^8$.

Q.2. Using De Moivre's theorem, find the least value of n ε N for which the expression : $(1+i)^n + (1 -i)^n$ is equal to $-2^{(n+2)/2}$.

Solution : 2

The given expression is :

$(1+i)^n + (1-i)^n$,

Let 1 + i = r(cos θ + isin θ) => r cos θ = 1; r sin θ = 1

Therefore, tan θ = 1 or θ = π/4 and r = √2.

Therefore, $(! + i)^{n} + (1 - i)^{n} = (r \cos \theta + i\, r \sin \theta)^{n} + (r \cos \theta - i\sin \theta)^{n}$

$= rn(\cos \pi/4 + i \sin \pi/4)n + rn(\cos \pi/4 - i \sin \pi/4)^{n}$

$= (\sqrt{2})^{n}\, [(\cos \pi/4\pi + i\sin \pi/4)^{n} + (\cos \pi/4 - i\sin n\pi/4)^{n}$

$= (\sqrt{2})^{n}[\cos n\pi/4 + i\sin n\pi/4 + \cos n\pi/4 - i\sin n\pi/4]$

$= (\sqrt{2})^{n} \times 2\cos n\pi/4$

$= 2^{(n/2+2)} \cos n\pi/4$

Or, $2^{(n/2+2)} \cos {}^{n}\pi/4 = -\, 2^{(n/2+2)}$

Or, $\cos n\pi/4 = -1 = \cos \pi$

Or, $n\pi/4 = \pi$

Therefore, $n = 4$.

Q.3. Using De Moivre's Theorem find the value of : $(2 - 2i)^{1/3}$.

Solution : 3

We have, $(2 - 2i)^{1/3}$.

Putting $2 = a \cos \theta$ and $-2 = a \sin \theta$ then $\tan \theta = -1$, $\theta = 135^{o}$.

$2 = a \cos 135^{o} = a\,(-1/\sqrt{2}) \Rightarrow a = -2\sqrt{2}$

Or, $(2 - 2i)1/3 = [-\, 2\sqrt{2}\, (\cos 135^{o} + i\sin 135^{o})]1/3$

$= [\, -2\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4)]1/3$

$= [\, (-\sqrt{2})3(\cos 3\pi/4 + i\sin 3\pi/4)]1/3$

$= (-\sqrt{2})3\times1/3[\cos \{(\, 2n\pi + 3\pi/4)\}/3 + i\sin \{(2n\pi + 3\pi/4)/3\}$

where , $n = 0, 1, 2$.

When $n = 0$, $(-\sqrt{2})[1/\sqrt{2} + i/\sqrt{2}] = -(1 + i)$

When $n = 1$, $(-\sqrt{2})[\cos 11\pi/12 + i\sin 11\pi/12]$

When $n = 2$, $(-\sqrt{2})[\cos 19\pi/12 + i\sin 19\pi/12]$

Q.4. Using De Moivre's Theorem, find the least value of n for which [(1 + √3i)/√2(1 + i)]n is purely imaginary.

Solution : 4

(1 + √3i)/√2(1 + i) = [(1 + √3i)(1 – i)]/[√2(1 + i)(1 – i)]

= [(1 + √3) + i(√3 – 1)]/2√2

Let (√3 + 1)/2√2 + i(√3 – 1)/2√2 = (r cos θ) + i(r sin θ)

Then, r cos θ = (√3 + 1)/2√2 and r sin θ = (√3 – 1)/2√2

Then, $r^2 = (r \cos \theta)^2 + (r \sin \theta)^2$

$= \{(\sqrt{3} + 1)/2\sqrt{2}\}^2 + \{(\sqrt{3} - 1)/2\sqrt{2}\}^2$

= (4 + 2√3)/8 + (4 – 2√3)/8 = 1

Therefore, r = √1 = ±1.

And tan θ= r sin θ/r cos θ = [(√3 – 1)/2√2]/[√3 + 1)/2√2]

= (√3 – 1)/(√3 + 1)

= (tan 60º – tan 45º)/(1 + tan 60º ×tan 45º)

= tan (60º – 45º) = tan 15º

Therefore, θ = 15º = π/12.

Hence, (1 + i√3)/√2(1 + i) = cos (π/12) + isin (π/12)

Therefore, [(1 + i√3)/√2(1 +i)]n = [cos (π/12) + isin (π/12)]n

= [cos (nπ/12) + isin (nπ/12)]

cos (nπ/12) isin (nπ/12) will be purely imaginary if and only if cos (nπ/12) = 0

Or, cos (nπ/12) = cos π/2 . cos 3π/2 , cos 5π/2 , ………..

Or, cos (nπ/12) = cos (6π/12), cos (18π/12), cos (30π/12) ………….

Therefore, nπ/12 = 6π/12, 18π/12, 30π/12, ………….

Or, n = 6, 18, 30, ……..

Therefore least value of n is 6.

Q.5. Using De Moivre's Theorem , find the least value of n for which the expression $\{(\sqrt{3} + 3i)/2\sqrt{3}\}^{2n-1}$ is purely real.

Solution : 5

We are given,

$\{(\sqrt{3} + 3i)/2\sqrt{3}\}2n - 1 = \{(1/2) + i(\sqrt{3}/2)\}^{2n-1}$

$= (\cos \pi/3 + i\sin \pi/3)^{2n-1}$

$= \cos (2n-1)\pi/3 + i\sin (2n-1)\pi/3$

The given expression is real, therefore its imaginary parts is zero. i.e.

$i\sin(2n-1)\pi/3 = 0$, but $i \neq 0$.

Therefore, $\sin(2n-1)\pi/3 = 0$

Therefore, value of $(2n-1)\pi/3 = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots\ldots\ldots$

When $(2n-1)\pi/3 = 0$, $2n - 1 = 0 \Rightarrow n = 1/2$.

This value is not acceptable as $n \in N$.

When $(2n-1)\pi/3 = \pi$, $2n - 1 = 3 \Rightarrow n = 2 \in N$.

When $(2n-1)\pi/3 = 2\pi$, $2n - 1 = 6 \Rightarrow n = 7/2$.

Which is again not acceptable as $n \in N$.

When $(2n-1)\pi/3 = 3\pi$, $2n - 1 = 9 \Rightarrow n = 5 \in N$.

Therefore, the least value of n is 2.

Q.6. Using De Moivre's Theorem, find the value of $(1 + i\sqrt{3})^4 + (1 - i\sqrt{3})^4$.

Solution : 6

We are given, $(1 + i\sqrt{3})^4 + (1 - i\sqrt{3})^4$

Let $r \cos \theta = 1$ and $r \sin \theta = \sqrt{3} \Rightarrow r \sin \theta / r \cos \theta = \sqrt{3}/1 \Rightarrow \tan \theta = \sqrt{3}$.

And $\theta = \pi/3$. also $r = \sqrt{\{(1)2 + (\sqrt{3})2\}} = \sqrt{(1 + 3)}\ \sqrt{4} = 2$.

Therefore, $(1 + i\sqrt{3}) = 2\,[1/2 + i\sqrt{3}/2] = 2\,[\cos \pi/3 + i\sin \pi/3]$

and $(1 - i\sqrt{3}) = 2[\cos \pi/3 - i\sin \pi/3]$

Hence, $(1 + i\sqrt{3})^{4} + (1 - i\sqrt{3})^{4} = (2)^{4}[\cos \pi/3 + i\sin \pi/3]^{4} + (2)^{4}[\cos \pi/3 - i\sin \pi/3]^{4}$

$= (2)^{4}[\cos^{4}\pi/3 + i\sin^{4}\pi/3 + \cos^{4}\pi/3 - i\sin^{4}\pi/3]$

$= (2)^{4}\times 2\cos^{4}\pi/3$

$=(2)5 \cos (\pi + \pi/3)$

$= (2)5 (- \cos \pi/3)$

$= (2)5 (- 1/2) = -16$.

Q.7. Using De Moivre's Theorem , find the least value of n for which the expression $[(1 + i)/(1 - i)]^{n}$ is purely imaginary.

Solution : 7

We are given, $[(1 + i)/(1 - i)]^{n}$

Let $1 = r\cos\theta$ and $1 = r\sin\theta$

Then $r^{2}(\cos^{2}\theta + \sin^{2}\theta) = 1 + 1 = 2 \Rightarrow r^{2} = 2 \Rightarrow r = \sqrt{2}$

And $\tan\theta = 1 \Rightarrow \theta = \pi/4$.

Hence, $1 + i = r(\cos\theta + i\sin\theta)$ and $1 - i = r(\cos\theta - i\sin\theta)$

And $(1 + i)n/(1 - i)n = [rn(\cos\theta + i\sin\theta)n]/[rn(\cos\theta - i\sin\theta)^{n}]$

$= (\cos\theta\ i\sin\theta)n\ (\cos\theta - i\sin\theta)^{-n}$

$= (\cos n\theta + i\sin n\theta)(\cos n\theta + i\sin n\theta)$

$= (\cos n\theta + i\sin n\theta)^{2} = \cos 2n\theta + i\sin 2n\theta$

As the given expression is purely imaginary, $\cos 2n\theta = 0$

Or, $\cos (2n\times\pi/4) = 0$, $\cos \pi/2 = 0$

Or, $\cos n\pi/2 = 0$, $n = \pm1, \pm3, \pm5, \pm7$ ……….

If we take only positive values of n then least value of n is 1.

Q.8. If $z = (13 - 5i)/(4 - 9i)$, prove by using De Moivre's Theorem that $z^6 = -8i$.

Solution : 8

We have, $z = (13 - 15i)/(4 - 9i)$

$= \{(13 - 15i)(4 + 9i)\}/\{(4 - 9i)(4 + 9i)\}$

$= 97(1 + i)/97$

$= 1 + i$

$= \sqrt{2}[1/\sqrt{2} + 1/\sqrt{2}i]$

$= \sqrt{2}[\cos \pi/4 + i\sin \pi/4]$

Therefore, $z^{\mathbf{6}} = \{\sqrt{2}[\cos \pi/4 + i\sin \pi/4]\}^{\mathbf{6}}$

$= 8[\cos 6\pi/4 + i\sin 6\pi/4]$

$= 8[0 - i] = -8i.$