Linear Differential Equation

Q.1. Solve the differential equation : $(x + 1)\, dy/dx - y = e^{3x} (x + 1)^2$.

Solution : 1

We have $(x + 1)\, dy/dx - y = e^{3x} (x + 1)2$.

Dividing both sides by (x + 1) , we get

$dy/dx - 1/(x + 1) = e^{3x} (x + 1)$.

Therefore, I. F. = $e - \int 1/(x + 1)dx = e - \log(x + 1) = 1/(x + 1)$. Multiplying by I. F. we get $1/(x + 1)\, [dy/dx - 1/(x + 1)] = e^{3x}. (x + 1) . 1/(x + 1)$.

ntegrating both sides , we get $y.1/(x + 1) = \int e^{3x}\, dx$ Or, $y/(x + 1) = (e^{3x})/3 + c$ Or, $y = \{(e^{3x}) /3\}(x + 1) + c (x + 1)$.

Q.2. Solve the differential equation : $ydx - (x +2y^2)dy = 0$.

Solution : 2

We have $ydx - (x + 2y^2)dy = 0$

Or, $ydx/dy - x - 2y^2 = 0$

Or, $dx/dy - x/y - 2y = 0$

Or, $dx/dy - x/y = 2y$

I.F. = $e^{\int pdy} = e - \int^{(1/y)dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = 1/y$.

Multiplying both sides by 1/y , we get

$1/y[dx/dy = x/y] = 1/y \times 2y$

Integrating , we get

$x.1/y = \int 2dy$

Or, $x/y = 2y + c \Rightarrow x = 2y^2 + cy$.

Q.3. Solve the differential equation : tan x dy/dx + 2y = sec x.

Solution : 3

We have tan x dy/dx + 2y = sec x

Or, dy/dx + (2/tan x)y = 1/sin x [dy/dx + Py = Q]

Here we have P = 2/tan x, Q = 1/sin x

I . F . = $e^{\int Pdx}$ = $e^{2}\int \cot x dx$ = $e^{2\log \sin x}$ = e^log $\sin^{2} x$ = $\sin^{2} x$

Integrating, we get y.$\sin^{2} x$ = ∫(1/sin x).$\sin^{2} x$ dx + c

Or, ($\sin^{2}$ x)y = ∫sin xdx + c = – cos x + c

Or, ($\sin^{2}$ x) y + cos x = c .

Q.4. Solve the differential equation : (1 – x^2) dy/dx – xy = x , given y = 2 when x = 0 .

Solution : 4

We have, (1 – x^2) dy/dx – xy = x

Or, dy/dx – [x /(1 – x^2)]y = x/(1 – x^2)

The I.F. = P = e^∫–x /(1 – x^2)dx

Then in ∫–x /(1 – x^2)dx [put 1 – x^2 = t => dt = –2xdx

= 1/2∫dt/t = 1/2 log t

Then P = e1/2log t = elog√t = √t = √(1 – x^2)

Therefore, dy/dx.√(1 – x^2) – [x/(1 – x^2)]y√(1 – x^2) = x/(1 – x^2)×√(1 – x^2)

Or, d/dx[y.√(1 – x^2)] = x/√(1 – x^2)

Integrating we get ∫d/dx[y.√(1 – x^2)].dx = ∫[x/√(1 – x^2)]dx + c

Or, y.√(1 – x^2) = 1/2 ∫[2x/√(1 – x^2)]dx + c

$= -1/2 \int[-2x/(1 - x^2)]dx + c$

$= -1/2\int dt/\sqrt{t} + c$ [where, $t = 1 - x^2$ => $dt = -2xdx$]

$= -1/2[\, t^{-1/2}] + c$

$= - \sqrt{(1 - x^2)} + c$

Or, $y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} = c$

Putting x = 0 and y = 2 we get c = 3

Hence, $y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} = 3$.

Q.5. Solve : $(1 + y + x^2y)dx + (x + x^3)dy = 0$.

Solution : 5

We have $(1 + y + x^2y)dx + (x + x^3)dy = 0$

Or, $1 + y + x^2y + (x + x^3).dy/dx = 0$

Or, $1 + y(1 + x^2) + x(1 + x^2).dy/dx = 0$

Or, $dy/dx + [y(1 + x^2)]/[x(1 + x^2)] + 1/[x(1 + x^2)] = 0$

Or, $dy/dx + y/x = -1/[x(1 + x^2)]$

Which is of the form : dy/dx + P.y = Q; P & Q are function of x.

Integrating factor $= e^{\int Pdx} = e^{\int(1/x)dx}$ [P = 1/x]

$= e^{\log x} = x$.

Multiplying both sides by I.F., we get

$x.[dy/dx + y/x] = x .\{-1/[x(1 + x^2)]\}$

Or, $x.dy/dx + y = -1 /(1 + x^2)$

Or, $d(x.y)/dx = -1/(1 + x^2)$

Integrating both sides, we get

$\int d/dx(x.y).dx = \int[(-1)/(1 + x2)]dx$

Or, x.y = tan –1 x + c

Q.6. Solve : $\cos^2 x$ dy/dx + y = tan x.

Solution : 6

We are given, $\cos^2 x$ dy/dx + y = tan x

Or, dy/dx +y.$\sec^2 x$ = tan x. $\sec^2 x$ [dividing by $\cos^2 x$]

This is of the form , dy/dx + Py = Q, here P = $\sec^2 x$

∫P.dx = ∫$\sec^2 x$ dx = tan x.

Integrating factor = $e^{\int P.dx} = e^{\tan x}$

Multiplying both sides by I.F., we get

etan x. dy/dx + y.$^{\tan x}$.$\sec^2 x$ = $e^{\tan x}$. tan x. $\sec^2 x$

Or, d/dx(y.etan x) = $e^{\tan x}$. tan x. $\sec^2 x$.

Integrating both sides we get,

∫d/dx(y. $e^{\tan x}$).dx = ∫$e^{\tan x}$.tan x.$\sec^2 x$. dx

Or, y.$^{\tan x}$ = ∫$e^{\tan x}$. tan x. $\sec^2 x$. dx

= ∫e^{t}.t.dt [put tan x = t => $\sec^2 x$ dx = dt]

= t∫e^{t}t.d^{t} – ∫e^{t}.d^{t} [Integrating by parts]

= t.e^{t} – e^{t} + c

= $e^{\tan x}$(tan x –1)

Therefore, the solution is y.$^{\tan x}$ = etan x(tan x – 1) + c.

Q.7. Solve : dy/dx + {2x/(1 + x^2)}y = 1/(1 + x^2)^2.

Solution : 7

We are given,

dy/dx + {2x/(1 + x^2)}y = 1/(1 + x^2)^2

This is of the form

dy/dx + Py + Q , where P and Q are functions of x.

P = 2x/(1 + x^2), then ∫Pdx = ∫2x/(1 + x^2)dx = log (1 + x^2)

Therefore, I.F. = e∫Pdx = e^log (1 + x^2) = 1 + x^2

Multiplying both sides by (1 + x^2), we get

(1 + x^2).dy/dx + 2x.y = 1/(1 + x^2)

Or, d/dx[y.(1 + x^2)] = 1/(1 + x^2)

Integrating both sides we get

∫d/dx[y.(1 + x^2)].dx = ∫dx/(1 + x^2) + c

Or, y.(1 + x^2) = tan −1 (x) + c.

Q.8. Solve : (1 + y^2) dx/dy = tan $^{-1}$ y – x .

Solution : 8

We are given , (1 + y^2) dx/dy = tan $^{-1}$ y – x

Or, dx/dy = (tan $^{-1}$ y)/(1 + y^2) – x/(1 + y^2)

Or, dx/dy + x.{1/(1 + y^2)} = tan $^{-1}$ y /(1 + y^2)

This is of the form :

dx/dy + Px = Q

∫P.dy = ∫dy/(1 + y^2) = tan $^{-1}$ y

I.F. = e∫Pdy = etan $^{-1}$y

Multiplying by I.F.,

we get e^tan $^{-1}$ y [dx/dy + {1/(1 + y^2)}.x] = e^tan $^{-1}$ y [tan^{-1} y/(1 + y^2)]

Or, d/dy[etan^{-1} y. x] = e^tan^{-1} y [tan^{-1} y/(1 + y^2)]

Integrating ,

we get e^tan^{-1} y . x = ∫e^tan^{-1} y [tan^{-1} y/(1 + y^2)] dy

Now ∫ e^tan^{-1} y [tan^{-1} y/(1 + y^2)]dy

[put y = tan θ => dy = sec^2 θ dθ and tan^{-1} y = θ]

= ∫ eθ .θ. (1/sec^{2}θ). sec^{2}θ. dθ

= ∫θ.eθ dθ = θ. eθ – ∫1.eθ. dθ

= θ.eθ – eθ = eθ.(θ^{-1})

= e^tan^{-1} y .(tan^{-1} y^{-1})

Therefore, x.e^tan^{-1} y = e^tan^{-1} y.(tan^{-1} y^{-1}) + c .

Q.9. Solve the following differential equation : sin x dy/dx – y = cos^2 x sin x tan (x/2).

Solution : 9

We are given,

sin x dy/dx – y = cos^2 x sin x tan (x/2)

Or, dy/dx – y/sin x = cos^2 x tan (x/2) [Dividing by sin x]

It is of the form, dy/dx + Py = Q ,

∫Pdx = ∫(–1/sin x) dx = ∫(–cosec x) dx = –log tan (x/2) = log cot (x/2)

I.F. = e∫Pdx = elog cot (x/2) = cot (x/2)

Multiplying by cot(x/2), we get

(dy/dx – y/sin x) cot(x/2) = cos^2 x tan (x/2)cot (x/2)

integrating , we get y cot(x/2) = ∫cos^2 x dx + c

= ∫[(1 + cos ^{2}x)/2]dx + c

y cot(x/2) = (1/2)x + (sin 2x)/4 = c .

Q.10. Solve the following differential equation for a particular solution : dy = (5x – 4y)dx ; when y = 0 and x = 0.

Solution : 10

We have, dy = (5x – 4y)dx

Or, dy/dx + 4y = 5x

I.F. = $e∫4dx = e^{4x}$

Multiplying by e4x and integrating we get,

$y.e^{4x} = ∫5x.e^{4x}\ dx$

$= 5x.e^{4x}/4 – 5/4∫e^{4x}\ dx + c$

$= 5/^{4x}.e^{4x} – 5/16\ e^{4x} + c.$