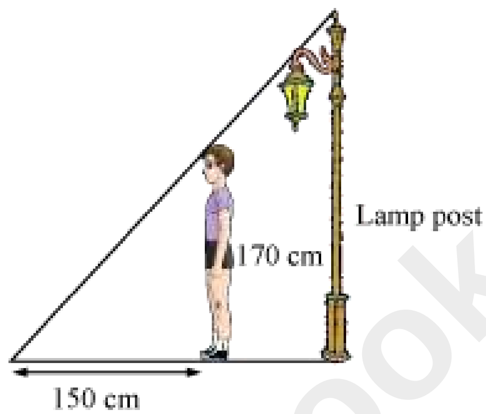
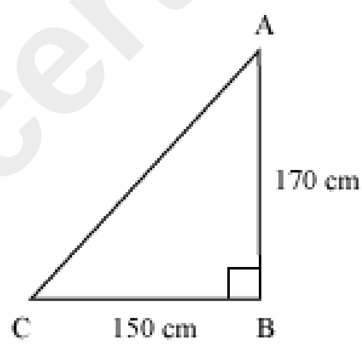
Trigonometrical Identities

Trigonometric Ratios

Suppose a boy is standing in front of a lamp post at a certain distance. The height of the boy is 170 cm and the length of his shadow is 150 cm.



You can see from the above figure that the boy and his shadow form a right-angled triangle as shown in the figure below.



The ratio of the height of the boy to his shadow is 170:150 i.e., 17:15.

Is this ratio related to either of the angles of ΔABC?

We can also conclude the following:

cos A $= \frac{1}{\sec A}$ **, tan A** $= \frac{1}{\cot A}$

Also, note that

tan A = $\frac{\sin A}{\cos A}$ **and cot A =** $\frac{\cos A}{\sin A}$

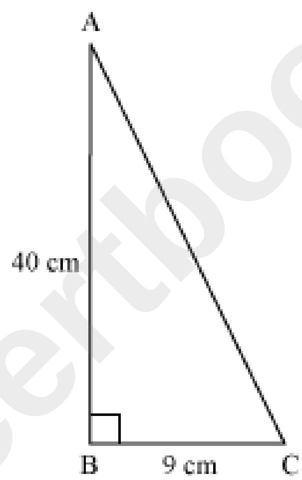
Let us now solve some more examples based on trigonometric ratios.

Example 1:

In a triangle ABC, right-angled at B, side AB = 40 cm and BC = 9 cm. Find the value of sin A, cos A, and tan A.

Solution:

It is given that AB = 40 cm and BC = 9 cm



Using Pythagoras theorem in ΔABC, we obtain

$(AC)^2 = (AB)^2 + (BC)^2$

$(AC)^2 = (40)^2 + (9)^2$

$(AC)^2 = 1600 + 81$

$(AC)^2 = 1681$

$(AC)^2 = (41)^2$

AC = 41 cm

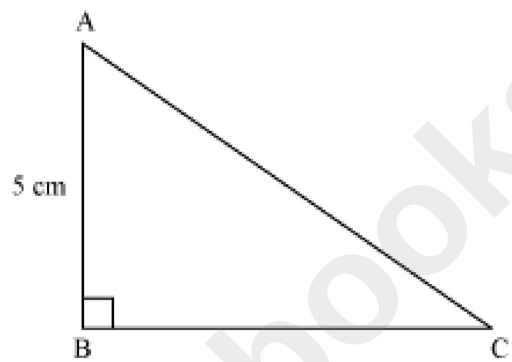
Now, sin A $= \frac{BC}{AC} = \frac{9}{41}$

cos A $= \frac{AB}{AC} = \frac{40}{41}$

tan A $= \frac{BC}{AB} = \frac{9}{40}$

Example 2:

From the given figure, find the values of cosec C and cot C, if AC = BC + 1.



Solution:

Now, it is given that AB = 5 cm and

AC = BC + 1 … (1)

By Pythagoras theorem, we obtain

$(AB)^2 + (BC)^2 = (AC)^2$

$\Rightarrow (AC)^2 - (BC)^2 = (AB)^2$

$\Rightarrow (BC + 1)^2 - (BC)^2 = (5)^2$ [Using (1)]

$\Rightarrow (BC)^2 + 1 + 2BC - (BC)^2 = 25$

$\Rightarrow 2BC = 25 - 1$

$\Rightarrow 2BC = 24$

$\Rightarrow BC = 12$ cm

$\therefore AC = 12 + 1 = 13$ cm

Thus, cosec C = AC/AB

= 13/5

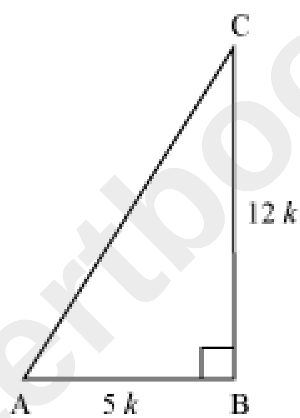
And, cot C = BC/AB

= 12/5

Example 3:

In a right-angled triangle ABC, which is right-angled at B, tan A = 12/5. Find the value of cos A and sec A.

Solution:



It is given that tan A = 12/5

We know that tan A = BC/AB

$\Rightarrow \frac{BC}{AB} = \frac{12}{5}$

Let BC = 12*k* and AB = 5*k*

Using Pythagoras theorem in ΔABC, we obtain

$(AC)^2 = (AB)^2 + (BC)^2$

$= (5k)^2 + (12k)^2$

$= 25k^2 + 144k^2$

$(AC)^2 = 169k^2$

$AC = 13k$

Now, cos A $= \frac{\text{Side adjacent to } \angle A}{\text{Hypotenuse}}$

$= \frac{5k}{13k}$

$= \frac{5}{13}$

Sec A $= \frac{1}{\cos A}$

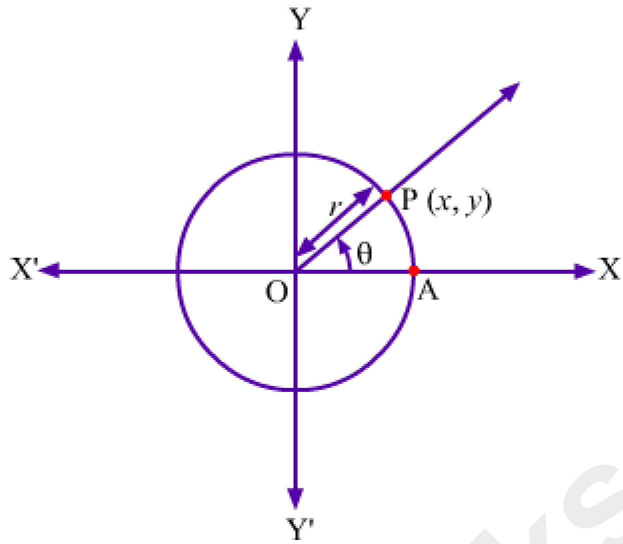
$= \frac{1}{\frac{5}{13}}$

$= \frac{13}{5}$

Trigonometric Identities

Now, let us prove these identities.

Let us take a standard circle with radius r such that it intersects the X-axis at point A. Also, let the initial arm OA is rotated in anti-clockwise direction by an angle ?



In the figure, the terminal arm intersects the circle at point P (x, y) where x, $y \neq 0$ and OP = r.

By the definition of trigonometric ratios, we have

$$\sin\theta = \frac{y}{r}, \qquad \operatorname{cosec}\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{x}{y}$$

Now, OP is a distance between origin O (0, 0) and point P (x, y) which can be obtained by distance formula as follows:

$$\text{OP} = \sqrt{(x-0)^2 + (y-0)^2}$$
$$\Rightarrow r = \sqrt{x^2 + y^2}$$
$$\Rightarrow r^2 = x^2 + y^2 \quad \ldots\text{(i)}$$

(1) On dividing both sides of the equation (i) by r^2, we get

$$\frac{r^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

$$\Rightarrow 1 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$

$$\Rightarrow 1 = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

From this identity, we get two results

I. $\sin^2\theta = 1 - \cos^2\theta$

II. $\cos^2\theta = 1 - \sin^2\theta$

(2) On dividing both sides of the equation (i) by x^2 ($x \neq 0$), we get

$$\frac{r^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$$

$$\Rightarrow \left(\frac{r}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$$

$$\Rightarrow \sec^2\theta = 1 + \tan^2\theta$$

$$\Rightarrow 1 + \tan^2\theta = \sec^2\theta$$

From this identity, we get two results

I. $\tan^2\theta = \sec^2\theta - 1$

II. $\sec^2\theta - \tan^2\theta = 1$

(3) On dividing both sides of the equation (i) by y^2 ($y \neq 0$), we get

$$\frac{r^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$$

$$\Rightarrow \left(\frac{r}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$$

$$\Rightarrow \operatorname{cosec}^2\theta = \cot^2\theta + 1$$

$$\Rightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

From this identity, we get two results

I. $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

II. $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Corollary:

(i) When $x = 0$ then we have

$\sin\theta = \frac{y}{r}$, $\quad \operatorname{cosec}\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r} = \frac{0}{r} = 0$, $\quad \sec\theta = \frac{r}{x} = \frac{r}{0}$ (Undefined)

$\tan\theta = \frac{y}{x} = \frac{y}{0}$ (Undefined), $\quad \cot\theta = \frac{x}{y} = \frac{0}{y} = 0$

In this case, the identities $\sin^2\theta + \cos^2\theta = 1$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ exist but the identity $1 + \tan^2\theta = \sec^2\theta$ does not exist.

(ii) When $y = 0$ then we have

$\sin\theta = \frac{y}{r} = \frac{0}{r} = 0$, $\quad \operatorname{cosec}\theta = \frac{r}{y} = \frac{r}{0}$ (Undefined)

$\cos\theta = \frac{x}{r}$, $\quad \sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x} = \frac{0}{x} = 0$, $\quad \cot\theta = \frac{x}{y} = \frac{x}{0}$ (Undefined)

In this case, the identities $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$ exist but the identity $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ does not exist.

Let us now solve some more problems using trigonometric identities.

Example 1:

Find the value of the expression ($\sec^2 27° – \tan27°.\cot63°$).

Solution:

$\sec^2 27° – \tan27°.\cot63° = \sec^2 27° – \tan27°.\cot (90° – 27°)$

[27° and 63° are complementary angles]

$= \sec^2 27° – \tan27°.\tan27°$ [cot (90° – ?) = tan ?]

$= \sec^2 27° – \tan^2 27°$

$= 1 + \tan^2 27° – \tan^2 27°$ [Using the identity $1 + \tan^2\theta = \sec^2\theta$]

= 1

Thus, the value of the given expression is 1.

Example 2:

Write all the trigonometric ratios in terms of sin A.

Solution:

Using the identity

$\sin^2 A + \cos^2 A = 1$,

we can write, $\cos^2 A = 1 – \sin^2 A$

Taking square root on both sides,

$\cos A = \sqrt{1-\sin^2 A}$ ...(i)

Now, $\tan A = \frac{\sin A}{\cos A}$

$= \frac{\sin A}{\sqrt{1-\sin^2 A}}$ [Using (i)]

$\Rightarrow \tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}}$ ...(ii)

$\cot A = \dfrac{1}{\tan A}$

$\Rightarrow \cot A = \dfrac{\sqrt{1-\sin^2 A}}{\sin A} \quad ...(iii)$

$\sec A = \dfrac{1}{\cos A}$

$\Rightarrow \sec A = \dfrac{1}{\sqrt{1-\sin^2 A}} \quad ...(iv)$ [Using (i)]

and, $\text{cosec} A = \dfrac{1}{\sin A} \quad ...(v)$

The trigonometric ratios in terms of sin A are given by (i), (ii), (iii), (iv), and (v).

Example 3:

Simplify the following expression.

[(1 + cot A – cosec A)(1 + tan A + sec A)]

Solution:

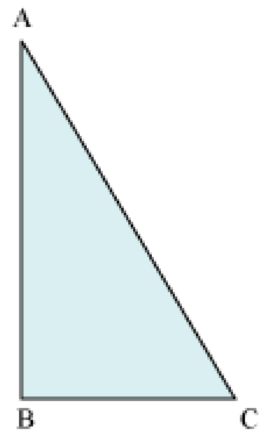
(1 + cot A – cosec A)(1 + tan A + sec A)

$$= \left(1+\frac{\cos A}{\sin A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{1}{\cos A}\right)$$
$$= \left(\frac{\sin A+\cos A-1}{\sin A}\right)\left(\frac{\cos A+\sin A+1}{\cos A}\right)$$
$$= \frac{(\sin A+\cos A)^2-1^2}{\sin A.\cos A}$$
$$= \frac{\sin^2 A+\cos^2 A+2.\sin A.\cos A-1}{\sin A.\cos A}$$
$$= \frac{1+2.\sin A.\cos A-1}{\sin A.\cos A} = \frac{2.\sin A.\cos A}{\sin A.\cos A} = 2$$

Thus, the value of the given expression is 2.

Trigonometric Ratios Of Complementary Angles

Consider the following figure.



Here, a right-angled triangle ABC has been shown. In this triangle, suppose that the value of sin C is 12/13.

Can we find the value of cos A?

We can use these relations for simplifying the given expression.

For example: Let us express **sec 55°− cosec 89°** in terms of trigonometric ratios of angles between 0° and 45°.

Now, how can we do so? Let us see.

Since 55° and 35° are complementary angles and also 89° and 1° are complementary angles, we can write 55° as (90° − 35°) and 89° as (90° − 1°).

Therefore,

sec 55° − cosec 89° = sec (90° − 35°) − cosec (90° − 1°)

= cosec 35° − sec 1°

[∵ sec (90° – A) = cosec A and cosec (90° – A) = sec A]

∴ sec 55° − cosec 89°= cosec 35° − sec 1°

Let us now solve some more examples involving trigonometric ratios of complementary angles.

Example 1:

Find the value of sin 53° – cos 37°.

Solution:

We know that 53° and 37° are complementary angles as

53° + 37° = 90°

∴ We can write 37° as (90° – 53°).

∴ sin 53° – cos 37° = sin 53° – cos (90° – 53°)

= sin 53° – sin 53°

[∵ cos (90° – A) = sin A]

= 0

Thus, the value of (sin 53° – cos 37°) is 0.

Example 2:

Evaluate $\frac{\textbf{cosec } 27^\circ}{\textbf{sec } 63^\circ}$

Solution:

Here, 27° and 63° are complementary angles as 27° + 63° = 90°

∴ We can write 27° = 90° – 63°

Now,

$$\frac{\text{cosec}27^\circ}{\sec 63^\circ} = \frac{\text{cosec}\left(90^\circ - 63^\circ\right)}{\sec 63^\circ}$$

$$= \frac{\sec 63^\circ}{\sec 63^\circ} \qquad \left[\because \text{cosec } \left(90^\circ - \text{A}\right) = \sec \text{A}\right]$$

$$= 1$$

Example 3:

Prove that tan 2A = cot 3A, when A = 18°.

Solution:

When A = 18°,

L.H.S = tan 2A = tan (2 × 18°)

= tan 36°

R.H.S = cot 3A = cot (3 × 18)

= cot 54°

54° and 36° are complementary angles.

∴ We can write 54° as 90° – 36°.

Therefore, cot 3A = cot 54°

= cot (90° – 36°)

= tan 36° [∵ cot (90° – A) = tan A]

∴ L.H.S = R.H.S = tan 36°

∴ tan 2A = cot 3A

Example 4:

If sin A = cos A, then prove that A = 45°.

Solution:

It is given that sin A = cos A

⇒ sin A = sin (90° – A)

[∵ sin (90° – A) = cos A]

⇒ A = 90° – A

⇒ 2A = 90°

$\Rightarrow A = \dfrac{90^\circ}{2}$

$\Rightarrow A = 45^\circ$

Hence, proved

Example 5:

If P, Q, and R are interior angles of a triangle PQR, which is right-angled at Q, then show that

$$\cot\left(\frac{P+R}{2}\right) = \tan\frac{Q}{2}$$

Solution:

Now, P, Q, and R are the interior angles of the triangle PQR. Therefore, their sum should be 180°.

∴ P + R = 180 – Q

Now, consider the L.H.S. = $\cot\left(\dfrac{P+R}{2}\right)$

= $\cot\left(\dfrac{180^\circ - Q}{2}\right)$

= $\cot\left(90^\circ - \dfrac{Q}{2}\right)$

= $\tan\dfrac{Q}{2}$ [∵ cot (90° – A) = tan A]

= R.H.S.

∴ $\cot\left(\dfrac{P+R}{2}\right) = \tan\dfrac{Q}{2}$

Hence, proved

Example 6:

Prove that

tan1°× tan 2°× tan 3° … tan 87°× tan 88°× tan 89° = 1

Solution:

Here, 1° and 89° are complementary angles as 1° + 89° = 90°

Therefore, we can write 89° = 90° – 1°

Similarly, 88° = 90° – 2°

87° = 90° – 3°

46° = 90° – 44° and so on

Now, the L.H.S is

tan 1° × tan 2° × tan 3° … tan 44° × tan 45° × tan 46° … tan 87° × tan 88° × tan 89°

= tan 1° × tan 2° × tan 3° … tan 44° × tan 45° × tan (90° – 44°) … tan (90° – 3°) tan (90° – 2°) tan (90° – 1°)

= tan 1° × tan 2° × tan 3° … tan 44° × tan 45° × cot 44° … cot 3° × cot 2° × cot 1°

[∵ tan (90° – A) = cot A]

$$= \tan 1° \times \tan 2° \times \tan 3° \ldots \tan 44° \times \tan 45° \times \frac{1}{\tan 44°} \ldots \frac{1}{\tan 3°} \times \frac{1}{\tan 2°} \times \frac{1}{\tan 1°}$$

$$\left[\because \cot A = \frac{1}{\tan A}\right]$$

= tan 45°

= 1

= R.H.S

∴ tan1° × tan 2° × tan 3° … tan 87° × tan 88° × tan 89° = 1

Hence, proved

Trigonometric Tables

We know the sine, cosine, and tangent values of angles such as 0°, 30°, 45°, 60°, and 90°.

However, what about angles such as 1°, 31°, 44°, 66°, and so on?

Do sine, cosine, and tangent values exist for these angles?

If yes, then how do we find these values?

Well, sine, cosine, and tangent values exist for all angles θ, where θ ∈ *R*

If we know the sine, cosine, and tangent values for θ lying between 0° and 90°, then we can find the trigonometric ratio values for all other angles using trigonometric identities and complementary angle identities.

To find the sine, cosine, and tangent values of all angles between 0° and 90°, we use the trigonometric tables.

Let us discuss some more examples based on trigonometric tables.

Example 1:

What is the value of $\sin 29°44'$?

Solution:

We read the table of natural sines in the horizontal line, which begins with 29°, and in the vertical column headed by $42'$.

sin 29°42' = 0.4955

Now, we read, in the same horizontal line, the value written in the mean difference column headed by $2'(44' = 42' + 2')$.

It is found to be 5.

As θ increases, sine value increases.

∴ The mean difference value is to be added.

$$\therefore \sin 29°44' = \sin(29°42' + 2')$$
$$= 0.4955 + 0.0005 = 0.4960$$

Example 2:

Using trigonometric tables, find the measure of angle θ when cos θ = 0.2473.

Solution:

Given: cos θ = 0.2473

$\cos 75°42' = 0.2470$

∴ Difference = 0.0003

Mean difference for $1'$ = 0.0003

$\because \theta = 75°42' - 1'$
$= 75°41'$

Therefore, the required angle is $75°41'$.

Example 3:

Using trigonometric tables, find the value of cosθ − sinθ when tan θ = 0.8977

Solution:

tan θ = 0.8977

tan $41°54' = 0.8972$

∴ Difference = 0.0005

Mean difference for $1' = 0.0005$

$\theta = 41°54' + 1' = 41°55'$

To find the value of cosθ − sinθ, we first find the value of $\cos 41°55'$ and sin $41°55'$.

$\cos 41°54' = 0.7443$

∴ Difference = $1'$

Mean difference for $1'$ = 0.0002

$\therefore \cos 41°55' = 0.7441$

$\sin 41°54' = 0.6678$

∴ Difference = $1'$

Mean difference of $1'$ = 0.0002

$\therefore \sin 41°55' = 0.6680$

$\cos 41°55' - \sin 41°55' = 0.7441 - 0.6680 = 0.0761$